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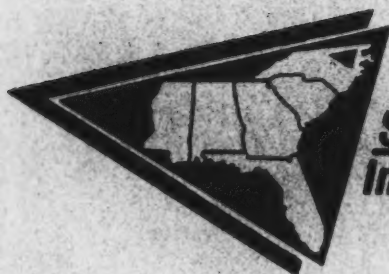
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ABSTRACT

This handbook is intended to contribute to the development of a theory of remedial instruction that can be easily understood and assimilated by teachers. The document evolved from research and workshop activity over a period of several years. It is designed for use by those with instructional or supervisory responsibilities in elementary schools, grades K-5. It is divided into three distinct and quite different sections. The first section describes four competencies consistently demonstrated by effective mathematics teachers in these grade levels. Those competencies are: (1) knowledge of the content and structure of mathematics; (2) knowledge of cognitive/developmental theories; (3) knowledge of and ability to diagnose students' errors; and (4) knowledge of and ability to select instructional strategies appropriate for reteaching mathematics skills and concepts. The second section is devoted to the diagnosis of specific errors frequently encountered in grades K-5. In addition, the development of strategies to effectively remediate these errors is included. The handbook concludes with an extensive primer on the concept of the training of trainers. (PK)

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EFFECTIVE MATHEMATICS TEACHING:

Remediation Strategies: Grades K-5

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Southeastern Educational Improvement Laboratory

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INTRODUCTION

The technological advances in our culture have resulted in an increased use of mathematics in daily life. In the past, emphasis was on "getting the right answer." In today's world, emphasis must be on mathematical literacy. Literacy in mathematics encompasses not only the ability to calculate, but competence in areas such as problem solving and estimating the reasonableness of results. These skills have applications as individuals use technology in their daily lives.

Yet even as we gain this new appreciation of the value of mathematical literacy, there is an apparent decline in performance in school mathematics throughout the country. The practical and emotional costs resulting from inadequate development of mathematical understanding are demonstrated by adults who have difficulty performing routine mathematical functions. In the classroom, repeated failure produces high levels of frustration and anxiety about mathematics that impede the teaching/learning process.

Until recently, a major assumption has been that poor performance in mathematics was a result of some type of learner deficiency. Females, especially, were often described as lacking "aptitude" for mathematics. One assumption has been that learning would improve if students would "work harder" or "care more about learning." Vast resources have been devoted to observing, classifying, and evaluating the way in which students learn mathematics. Clearly, students do have a responsibility to learn. It appears that teachers, in an attempt to remediate poor performance, must examine factors that could have contributed to the inadequate learning of mathematics. It has been suggested that failure to learn mathematics may be due to various causes other

than the students themselves. Failure to learn mathematics by many students may be due primarily to inferior teaching methods.

As reported by the National Science Foundation (1979), case study observers document the crucial role of the classroom teacher in effectively promoting math education. Results of a decade of research by the Missouri Mathematics Project (Good, Crouws & Ebmeirr, 1983) confirm that teacher knowledge about mathematics and knowledge about the way students learn place predictable limits on what a teacher can accomplish. In An Agenda For Action (1980), the National Council for the Teachers of Mathematics delineated specific mathematics skills that should be emphasized by teachers during the decade. Their recommendation was that emphasis be placed on the teaching of all strands of mathematics (when appropriate), as well as integrating the teaching of estimation, problem solving, and critical thinking skills into the mathematics curriculum. In addition, this document encourages the appropriate use of calculators and computers in the mathematics classroom. It recommends that the preservice as well as the in-service training of teachers be directed toward these goals.

Another event that has instigated a change in the role of the teacher is the mandate for minimum competency testing by many states. As a consequence of such testing, it has become necessary for teachers to provide remediation for those who do not demonstrate minimal learning skills. Classroom teachers have been, overnight, expected to become remedial teachers without the benefit of training in this area. There is, therefore, an increasing need for a theory of remedial instruction that can be easily understood and assimilated by teachers.

This handbook, Effective Mathematics Teaching: Remediation Strategies-Grades K-5, is offered as one contribution to the development of such a theory. It has evolved from research and workshop activity over a period of several

years sponsored by the Southeastern Educational Improvement Laboratory and its predecessor organization, the Southeastern Regional Council for Educational Improvement. The handbook is designed for use by those with instructional or supervisory responsibilities in elementary schools, grades K through 5. It is divided into three distinct and quite different sections. The first section describes four competencies consistently demonstrated by effective mathematics teachers in these grade levels, as reported in the literature--particularly in situations requiring remedial or corrective instruction. Those competencies are:

1. Knowledge of the content and structure of mathematics, i.e., the concepts, skills, and applications that constitute the discipline.
2. Knowledge of cognitive/developmental theories as they apply to the teaching and learning of mathematics.
3. Knowledge of and ability to diagnose students' errors in mathematics tasks.
4. Knowledge of and ability to select instructional strategies appropriate for reteaching mathematics skills and concepts.

This handbook's first section offers a fairly comprehensive discussion of these four competencies, providing information on the content and structure of math, cognitive and developmental learning theories, approaches to the diagnosis of learning problems, and an overview of instructional strategies.

Section II of this handbook is devoted to the diagnosis of specific errors frequently encountered in the six strands in grades K-5. In addition, the development of strategies to effectively remediate these errors is included. The section also includes instructional materials dealing with strategies on mental arithmetic, computational estimation, and problem solving. Finally, in

Section III, the handbook concludes with an extensive primer on the concept of the training of trainers. The training components include basic and generic information that may prove of particular value to the beginning trainer.

**COMPETENCIES
OF AN
EFFECTIVE
MATHEMATICS
TEACHER**

COMPETENCIES OF AN EFFECTIVE MATHEMATICS TEACHER

In attempting to enhance mathematics instruction, having an understanding of the competencies that are consistently demonstrated by an effective mathematics teacher is essential. While teaching is both an art and a science, it incorporates techniques, strategies, and approaches for learning. These strategies have been classified into four broad competencies that were synthesized from current research in mathematics education.

Knowledge of the Content and Structure of Mathematics

The effective teacher can:

1. Identify the strands of mathematics generally taught in the elementary school curriculum: numeration and concept of number, computation, functions and relations, measurement, geometry, and probability and statistics.
2. Relate subtopics in mathematics to topics previously taught and to topics to be presented in later instruction.
3. Describe the curricular sequencing of a variety of basal text series, including the major concepts taught at each grade level.

Knowledge of Cognitive/Developmental Theories

The effective teacher can:

1. Describe theories of cognitive development as applicable to the teaching and learning of mathematics.
2. Identify differences between the behavioral and cognitive/developmental theories in the teaching of mathematics.

3. Identify the four functional levels of the learner as applicable to the strands of math.

Knowledge of Diagnosis

The effective teacher can:

1. Demonstrate various diagnostic procedures that are most effective in assessing the cause of errors in the student's thinking, such as the analysis of written products and conducting of unstructured interviews.
2. Describe the difficulty in determining conceptual versus procedural errors.

Knowledge of Instructional Strategies

The effective teacher can:

1. Demonstrate a variety of instructional strategies so that individual learning characteristics can be considered in instructional activities.
2. Demonstrate a variety of instructional strategies for various subtopics of mathematics so that the diagnosis of a student error can be associated with a number of instructional options.
3. Use activities that increase the student's understanding and use of mathematical language.
4. Simplify most mathematics tasks into a manipulation of objects or an illustration model, using either teacher-made or commercial materials.

5. Describe and demonstrate the appropriate use of computers and calculators in the mathematics classroom.
6. Promote a positive attitude toward mathematics to students.

**KNOWLEDGE
OF THE
CONTENT AND
STRUCTURE
OF MATHEMATICS**

KNOWLEDGE OF THE CONTENT AND STRUCTURE OF MATHEMATICS

Strands of Math

Mathematics concepts commonly taught in grades K-5 are often grouped by math educators into strands or threads of math. The strands of math to be discussed in this handbook are:

1. Numeration

A. Prenumber concepts

1. Classifying
2. Sorting
3. Ordering
4. One-to-One Correspondence
5. Sets

B. Number Concepts

1. Counting
2. Ordinal and Cardinal Numbers
3. Face and Place Value
4. One-to-Many Correspondence
5. Few-to-One Correspondence
6. Concept of Zero
7. Bases and Systems
8. Estimation and Rounding

2. Computation

A. Addition, Subtraction, Multiplication, and Division and Their Properties

1. Whole Numbers
2. Rational Numbers
3. Decimals and Percents

B. Estimation of the Computation of:

1. Whole Numbers
2. Rational Numbers
3. Decimals and Percents

3. Relations and Functions

- A. Greater Than, Less Than
- B. Ordered Pairs, Graphing

4. Measurement

- A. Length: inches, feet, yards, miles, millimeters, centimeters, meters, kilometers
- B. Time
- C. Money
- D. Volume and Capacity: cups, pints, quarts, gallons, milliliters, liters
- E. Perimeter and Area
- F. Temperature: Fahrenheit, Celsius
- G. Weight and Mass: ounces, pounds, tons, grams, kilograms
- H. Estimation of Measurement

5. Geometry

- A. Solid Shapes
- B. Plane Shapes
 - 1. Lines
 - 2. Segments
 - 3. Rays
 - 4. Angles
- C. Congruency
- D. Symmetry

6. Probability and Statistics/Grouping

- A. Bar, Line, Pictographs
- B. Chance Occurrence
- C. Averaging (mean and median)
- D. Data Collection and Organization Predicting

7. Application of the Strands in Problem Solving

- A. Numeration
- B. Computation
- C. Relations and Functions
- D. Measurement
- E. Geometry
- F. Probability, Statistics, and Graphing

Structure of Mathematics

Structure is defined as the arrangement of all the parts into a whole. Structure has been described in various ways by basal series' scope and sequence charts, mathematical taxonomies and hierarchies, and concept cluster charts. The Checklist of Mathematical Concepts (1975) developed by the Kent State University Mathematics Education team, for example, provides an extremely comprehensive overview of the structure of mathematical concepts. The scope and sequence chart that accompanies the textbook teachers use and the curriculum guide provided by the school district and/or state will be useful to the classroom teacher.

Use of a structural framework such as a scope and sequence chart provides the teacher with a means of identifying skills that students might be expected to know at a given time and in the future. The effective mathematics teacher has comprehensive knowledge of: mathematical concepts or topics, the sequence in which concepts are usually taught, the frequency of exposure to concepts in the curriculum, and ways that simpler concepts relate to those that are more complex. This knowledge enables the teacher to identify the skills and concepts that need to be taught. The "big picture" of the broad scope of mathematics provides a conceptual framework for defining the learning needs of individual students.

An examination of scope and sequence charts reveals an interrelatedness among concept clusters. A student typically does not study and learn all of the ideas under one concept cluster and then proceed to the next cluster. Rather, there is a hierarchical scheme within each cluster, and student learning is reinforced through curriculum spiraling. Using a scope and sequence chart or another form that identifies expected learning outcomes for

each student is helpful in documenting the acquisition of math skills throughout the elementary school years.

One approach to the conceptualization of the structure or hierarchy of a particular concept is task analysis. This is the process of dissecting or breaking down a skill or concept into its component parts. The use of task analysis enables the teacher to be aware of the skills necessary for performing a particular mathematical task. The use of task analysis in facilitating the process of diagnosing student errors will be discussed later in the handbook.

Summary

The effective mathematics teacher has a clear understanding of the content and structure of his/her subject as it is commonly taught in grades K-5. The teacher is familiar with the six strands of math at these grade levels (concepts of number and numeration, computation, relations and functions, measurement, geometry, and probability/statistics). Further, the teacher understands the importance of teaching these concepts as interrelated skills. Finally, the effective mathematics teacher is able to use task analysis as a tool to understand the needs of individual students.

**KNOWLEDGE
OF
COGNITIVE/
DEVELOPMENTAL
THEORIES**

KNOWLEDGE OF COGNITIVE/DEVELOPMENTAL THEORIES

To improve instruction in mathematics, teachers need to understand the theories on which the subject is based. Instructional practices are currently influenced by two major theoretical orientations: the behavioral and the cognitive/developmental. The behavioral approach to learning and instruction has dominated the field of mathematics and education, in general, for some time. However, educators have become increasingly aware of developmental influences on a student's ability to learn. To clarify classroom practices generated by these approaches, it is useful to compare the theoretical assumptions underlying each theory.

Behavioral vs. Cognitive/Developmental Learning Theories

Behaviorists believe that learning is segmented, that it is a hierarchical sequential acquisition of knowledge and discrete skills. This view emphasizes that knowledge is a learned response to a given stimuli. A teaching strategy of a behaviorally oriented teacher is to present or "teach" a skill with the expectation that the student will learn the skill through repetitious practice. A typical remedial strategy would be to show the student how to perform the skill again "in a louder voice" or by "writing bigger on the chalkboard"--and then assign more problems for the student to practice. Emphasis in this approach is "how to": how to get the correct answer. Little attention is paid to sharing why a certain skill might be beneficial to the student.

In cognitive/developmental theory, learning is viewed as a change in the student's cognitive structure. After learning, a student "understands differently." This theoretical perspective also considers a child's developmental

and functional level as learning activities are developed for the classroom. Students' answers to mathematical tasks are viewed as keys for understanding developmental levels so that appropriate learning activities can be introduced. Further, the cognitive/developmental approach indicates that students are more likely to learn, when being introduced to a skill or concept, if they understand why it is in the study plan or "what it's for." Establishing the meaningfulness of the concept/skill being taught helps students understand how it can be useful to them.

The philosophy of teachers who embrace the behavioral view leads to different types of learning experiences for students. This handbook endorses and is built upon the cognitive/developmental approach to learning. The fact that elements in other learning theories offer ideas and strategies of value is not dismissed. For example, practicing a skill--a behaviorist construct--is an essential step in learning that skill.

Functional Levels of the Learner

In the cognitive/developmental approach to the teaching of mathematics, the developmental level of students is an important consideration when planning learning activities. Since learning in mathematics generally proceeds from the concrete to the abstract (Ausebel, 1978; Piaget, 1973), functional levels of learning have been identified that help the teacher determine appropriate classroom instruction. These functional levels are described as concrete, semiconcrete, semiabstract, and abstract. The learning of math concepts proceeds on a continuum from concrete to abstract understanding. Thus, teaching strategies need to address this broad range of mathematical thinking.

The concrete level is exemplified by the manipulation of objects in the students' environment. Through the manipulation of objects, conceptual understanding is acquired (Piaget, 1973). In the earlier grades and with remedial students in any grade, the use of hands-on materials is essential (Clayton, 1986; Engelhardt, Ashlock, and Wiebe, 1984; Good, Grouws, and Ebmeirer, 1983; Kamii, 1980, 1982, 1985). As they prepare lesson plans in mathematics, classroom teachers need to ask: How can I demonstrate this concept so that it is meaningful to students? How can the student touch, use, or manipulate objects that will help him/her develop a more comprehensive understanding of the concept? Manipulatives that facilitate learning are available through many commercial companies and include items such as geoboards, Cuisenaire rods, and Unifix cubes. Many effective manipulative materials that provide concrete experiences for children can be found in the daily environment, such as beans, buttons, toothpicks, coins, bottle tops, and many, many more.

The semiconcrete functional level of the learner employs the use of pictures of concrete objects in understanding mathematical skills and concepts. It is assumed that the concrete experiences students have with real-world objects provide the mental foundation so that they can visualize the concept through the use of pictures or models. This semiconcrete level of thinking and communication is found in the pictorial pages of kindergarten and elementary grade mathematics texts and materials (Kamii, 1985). While at the semiabstract functional level, the student builds knowledge based on the representations from the semiconcrete level. Strategies used at this level include tallying and the use of finger-math. These representations from the first mathematicians in history are still valuable techniques to use in representing concepts (Ausebel, 1978; Piaget, 1976; Scheer, 1980). The more complicated finger

manipulations of Chisanbop are another example of the complexity of thought that can encourage students to organize and communicate in mathematics at the semiabstract level. This developmental level can also allow students to build conceptual knowledge to prepare for abstractions. Many educators classify the semiconcrete and semiabstract levels together, since they are similar in nature in that they bridge the gap between the concrete and the abstract. For the purposes of this handbook, these levels are identified separately.

The most sophisticated of the functional levels of the learner is the abstract level: the use of mathematical symbols and numbers to describe relationships. Being able to understand and perform mathematical tasks using symbols requires students to first understand these tasks at the concrete level.

Students must have repetitive experiences with objects before they can see pictures of objects and recognize the concept of number. At a more advanced level, students must understand that symbols such as tally marks can represent pictures of objects as well as numerals. Only then will a specific numeral have meaning for them. In other words, the concept of "two-ness" must be developed before the abstract numeral 2 can be understood.

Functional Levels of the Learner

<u>Skills</u>	<u>Concrete Level</u>	<u>Semiconcrete Semiabstract</u>	<u>Abstract Level</u>
Adding Whole Numbers	Using Manipulative Objects To Show "Putting Together"	Using Picture Representations Of Objects	Using Number Sentences To Represent Concept
	$\begin{array}{ccc} 0 & 0 & \\ 0 & 0 & 0 \end{array}$	$\begin{array}{ccc} 0 & 0 & \\ 0 & 0 & 0 \end{array} \quad \cancel{1111}$	$4 + 1 = 5$
	Actual Objects	Picture Tally	

Summary

Knowledge of the cognitive/developmental and behavioral theories provides insight into how learning takes place and information that teachers can use to enhance their effectiveness in the mathematics classroom. Students will learn at an optimal level when they understand the purpose of the task to be learned or how it relates to their daily life. A primary skill of the effective mathematics teacher is to determine the functional level at which a student can best understand the concept being presented. Students of the same chronological age (Piaget, 1970, 1973) may vary widely in their background of mathematical experiences and the amount of remediation occurring in those experiences. As a result, the effective teacher recognizes that even older students may acquire better understanding if the conceptual basis is built through the functional levels with concrete, semiconcrete, semiabstract, and abstract experiences.

**KNOWLEDGE
OF
DIAGNOSIS**

KNOWLEDGE OF DIAGNOSIS

Use of the word "diagnosis" in education derives from a medical perspective that assumes that knowledge of the etiology of disease is necessary before a remedy can be prescribed. Influenced by cognitive/developmental theory, Schultz & Strawderman (1980) define diagnosis as "an examination of the learner's structures, the structures of the mathematics content which is being learned, and finally a review of the nature of the instructional structures currently in practice."

If the student is obtaining incorrect answers consistently, the teacher should "diagnose" the problem prior to trying to remediate the deficient skill. The importance of the teacher's role in diagnosis of student errors has been documented by several researchers. Two skills have been described by Engelhardt and Wiebe (1978) as basic to the diagnostic/remedial process:

1. the ability to identify correctly the mathematical learning difficulty that the child is experiencing; and
2. the ability to design instruction appropriate to the child's problem.

Leaders in the field of mathematics education suggest that diagnosis should not be limited to slow learners or underachievers, but rather should be available for all children, and that efforts should be geared toward preventive measures as well as corrective measures. For the classroom teacher to fulfill this role, he/she must have an understanding of a diagnostic model and the procedures used in the educational setting.

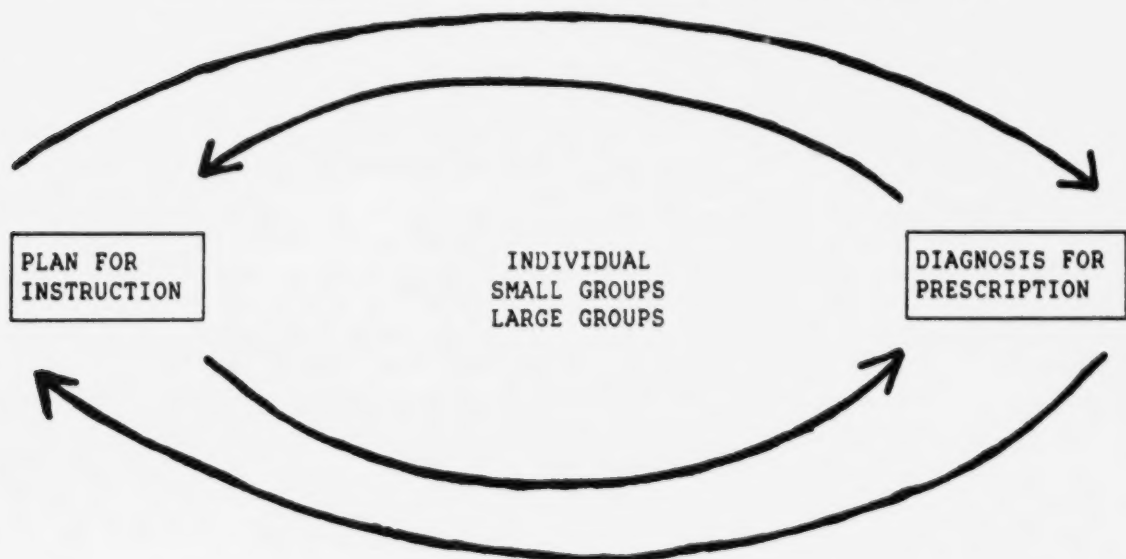
What are symptoms of faulty learning in mathematics? As early as 1935, Brueckner identified a number of important concerns: the inability to work three or four examples of a type of problem correctly, failure to improve with

practice, confusion of mathematics processes, and inability to apply what has been learned in practical situations.

The Diagnostic-Prescriptive Model

The diagnostic-prescriptive method is a model of teaching that assesses the errors that students make and subsequently structures the learning process so that these errors will be eliminated (West, 1981). In this method, testing is used to determine the direction of instruction and to assess the progress of the learner in obtaining the desired goals or concepts. Based on the results of testing, units of instruction are planned that build upon the knowledge and skills that a student already possesses.

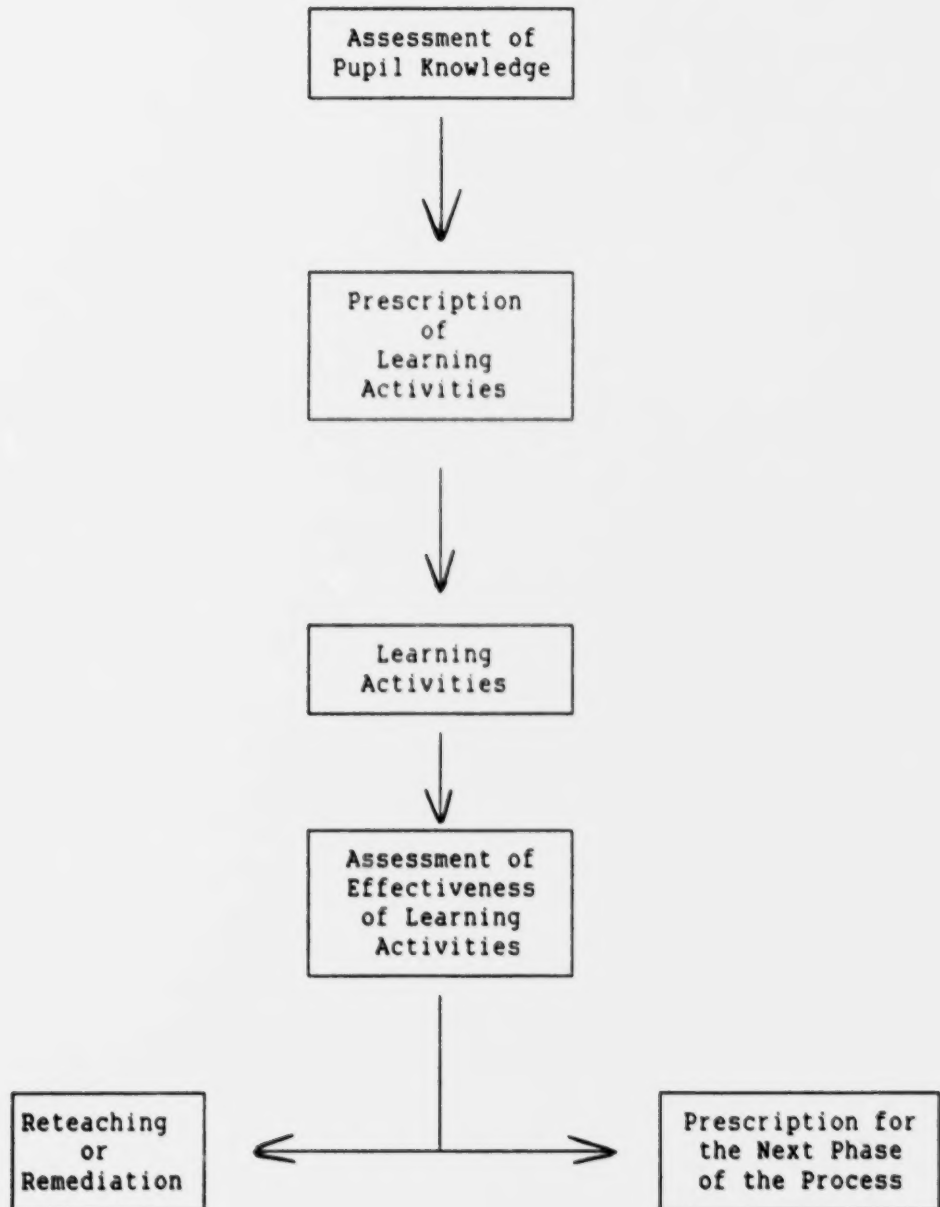
THE TEACHING CYCLE MODEL (Schultz and Strawderman, 1980)



Another way of conceptualizing this process is by viewing it as a continuous teaching-learning process (Clayton, 1986). Many teachers already use

this model of teaching. For them, learning to diagnose mathematical errors effectively is merely an addition to a familiar teaching model.

THE CONTINUOUS TEACHING-LEARNING PROCESS



Procedures for Diagnosis

Currently, two procedures are frequently used in diagnosing errors in mathematics: analysis of students' written products and analysis of thinking processes using oral interview techniques.

Analysis of a student's written work is a process of examining error patterns in any of the strands of math in order to determine points of breakdown in the learning. Engelhardt, Ashlock and Wiebe (1984) identified four general types of errors in computation.

1. Careless: incorrect responses resulting from responding without fully engaging in the task.

2. Mechanical: incorrect responses resulting from perceptual-motor difficulties such as misformed or misaligned symbols.

$$\begin{array}{r} 46 \quad 17 \\ - 3 \quad 114 \\ \hline 13 \quad + 25 \\ \hline 309 \end{array}$$

3. Conceptual: absent or misunderstood concept/principles, such as inappropriate concepts of zero or regrouping in subtraction.

$$\begin{array}{r} 67 \quad 16 \\ -29 \quad \times 4 \\ \hline 42 \quad 124 \end{array}$$

4. Procedural: incorrect responses resulting from misordered or inappropriate procedures, such as subtracting the minuend from the subtrahend.

$$\begin{array}{r} 30 \quad 13 \\ -17 \quad + 4 \\ \hline 42 \quad 8 \end{array}$$

While these researchers delineated these four types of errors in the computational strand, it appears they also apply to the other strands of math. It is important to note that errors should not be treated as separate difficulties. Rather, the teacher should look for patterns of errors that could be

used in remediating students' deficiencies.

The second procedure, an oral interview, is often used in conjunction with analysis of written products to provide information about thought processes underlying the work of students. It provides an opportunity for a teacher and a student to discuss thinking patterns in mathematics. An interview generally consists of a series of unstructured questions.

In such an interview, the teacher simply asks students to explain how they thought through the problem and obtained the responses given. Since the interview process is not commonly used in most mathematics classrooms, it is not unusual for students to find "talking about thinking" difficult. Frequent experiences with interviews and good teacher skills in questioning help students develop the ability to articulate their thoughts. Younger students may be expected to experience more difficulty in answering interview questions, since talking about one's thinking is an advanced cognitive skill. It may be more beneficial if the teacher asks the younger student to draw pictures or manipulate objects to demonstrate how a response was obtained.

Janet K. Scheer in "The Etiquette of Diagnosis" (1980) suggests the following guidelines to be followed for successful diagnosis in an oral interview setting:

1. Be accepting of the child's failures as well as the child's right answers. A teacher's body language and nonverbal communication play an important part.
2. Don't cue the child when he/she gives an incorrect answer, even when you know that the child knows the right answer.
3. Do be thorough; gain as much information as possible.

4. Don't interrupt a student when he/she makes an error. The temptation to correct students' errors is frequently difficult to resist. We must remember, however, that our immediate aim is to find out what the errors are and why the child makes them.
5. Don't hurry the student.
6. Avoid answering direct questions such as, "Was that right?" or "How am I doing?" A standard non-ego-damaging response, which usually satisfies the child, is, "You're doing fine."
7. Don't lead the child through a series of questions that will ultimately lead to the response for which you are looking. "Why did you say that?" or "Why did you do that?" asked in an easy conversational tone of voice will elicit the thought patterns you're after.
8. Ask the child to "think out loud."
9. Repeat the child's answer. This serves a twofold purpose. First, because many children have a tendency to speak softly or mumble when responding, we need to clarify what we think we have heard. Second, it gives us something factual to say that is neither positive nor negative and therefore will not influence the outcome of the diagnosis. (One thing to beware of is voice intonation--the tone of voice that says, "Are you sure?" and nonverbal cues, such as a raised eyebrow).
10. Encourage children to cross out and start over, rather than to erase, when they are working through written exercises.
11. Make notes of behavior patterns. Observation plays an important role in diagnosis. Squinting, nail-biting, fidgeting, sitting on hands (may indicate counting on fingers), daydreaming, and gazing out the

window may all imply things about the child that will provide useful information in a complete diagnosis.

Summary

The ability to diagnose student errors is essential for an effective mathematics teacher. Two strategies are generally used by teachers in the diagnostic process: diagnosis of written products (tests and classroom work) and oral interviews. Diagnosing errors enables a teacher to choose strategies for remediation that are appropriate for the cognitive/developmental level of students.

**KNOWLEDGE
OF
INSTRUCTIONAL
STRATEGIES**

KNOWLEDGE OF INSTRUCTIONAL STRATEGIES

The effective mathematics teacher recognizes the functional levels of students, uses diagnostic techniques to select instructional strategies, and incorporates knowledge of individual learner characteristics in the classroom. By using these strategies, the teacher is likely to better meet the individual needs of students. While the teacher may rely on newer mathematical tools such as calculators and computers, the emphasis must be on the acquisition of mathematics as a language and on the conceptual understanding of mathematics. To achieve these outcomes, the teacher's positive attitude toward mathematics is fundamental.

To select an appropriate strategy for teaching a skill to a student, a teacher needs a wide knowledge of several methods for teaching related concepts. Theories of the teaching-learning process in mathematics suggest that an effective initial step in this process is to provide instruction that is meaningful to and can be understood by the student. A second step is to provide drill-and-practice activities to "fix" the concepts previously learned (Brownell, 1935). Although many teachers have been taught that using drill and practice with skills that are to be learned or relearned will be sufficient to ensure student learning, this is not usually true. Meaningful initial or remedial instruction should include more than drill-and-practice activities. However, it is important that the practice of the skill not be done until the skill has been accurately or correctly learned. Otherwise, the student will practice incorrect learning. The instructional strategies chosen should be taught or retaught in a way that is meaningful to students and then practiced until the skill is thoroughly learned.

Use of Diagnosis

In selecting activities, both initial and remedial instruction should incorporate knowledge of the functional levels of the learner. Diagnosis of the student's errors will indicate the functional level of that student as well as where the breakdown in learning has occurred, i.e., what skill or concept has not been correctly learned. A scope and sequence chart can be used to determine the prerequisite skills necessary for instruction. The use of diagnosis in pinpointing a specific area of weakness enables the teacher to reteach the skill more efficiently, since information already known by the student need not be retaught.

Individual Learner Characteristics

The effective teacher who ascribes to theories of learning that emphasize the uniqueness of each individual learner (i.e., the cognitive/developmental perspective) seeks to know as much as possible about personal characteristics that seem to affect learning. Knowledge about such variables as intelligence, attitudes, social and economic background, sex, cognitive type, and anxieties is found in both the literature of psychology and the literature of mathematics education. Knowledge of past research may not simplify identification of each learner's uniqueness. It does, however, produce an awareness and flexibility for the effective teacher. The expectation that all learners of the same age or grade should perform alike changes to the expectation that individual differences in learning will challenge a teacher's diversity and creativity in instruction.

Math as a Language

Included in the various activities and materials suggested for effective mathematics teaching is a variety of speaking, reading, writing, and translating activities to develop mathematics as a language. Many teachers are introducing math as a language within the traditional mathematics curriculum. Strategies are chosen that enable students to relate mathematics to "real-world" situations in their everyday experiences, not just in the mathematics classroom. Simple activities include finding mathematical statements in other textbooks including history, literature, etc. Newspapers are an excellent source for teaching mathematics (Baron, London, Matchette, 1982). Activities such as using the advertising section for computation and word problem skills can be utilized. These simple strategies remove, or at least reduce, the abstractness of mathematics for students and promote an understanding of "why" mathematics is important for them.

Mathematical language, like all language, requires reading and writing skills. The effective teaching of mathematics includes generating reading problems, mathematics vocabulary development, and written expression activities. The teacher who asks students to explain mathematical words in their own language, not memorized definitions from the mathematics text, gains insight into how well such terms are understood. The effective mathematics teacher uses many activities from the language arts curriculum to facilitate students' development of mathematical language.

Materials and Activities

The effective mathematics teacher provides diverse materials and activities to meet the individual needs of each student, regardless of their functional level. Although the mathematics textbook may be used by the classroom teacher as one source of instructional materials, many additional ideas and activities can be obtained from products that are commercially available. If the mathematics classroom becomes a laboratory for experimentation and concrete "hands-on" activities involving collecting data and making graphs, students will be more likely to enjoy and understand mathematics. This handbook contains activities from each of the six strands of mathematics, suitable for use with children grades K-5. All of these activities can be easily and inexpensively conducted by the classroom teacher.

Use of the Calculator

Many mathematics teachers do not permit use of calculators in the classroom because they are not sure when their use is appropriate. Further, they are unsure about how to use calculators as instructional tools. The primary fear associated with the use of calculators is that this practice may engender a declining emphasis on development of basic skills (Suydam, 1982). Do calculators threaten basic skills? The answer to that question appears to be no, provided that the basics have first been developed with paper and pencil (Suydam, 1979). Many teachers appear to be unaware of this research. Specific research findings based on a synthesis of 79 studies (Hembree & Dessart, 1986) offer these conclusions:

1. Students who use a calculator in grades K-12 in conjunction with traditional instruction maintain their paper-and-pencil skills (basic

skills) without harm. Further, the use of calculators can improve the average student's basic paper-and-pencil-skills, both in basic operations and in problem solving.

2. Students using calculators possess somewhat better attitudes toward mathematics and an especially better self-concept in mathematics than noncalculator students. This statement applies across all grade and ability levels.

Other research suggests that students should not be required to compute large multidigit problems by paper and pencil. If the students have understood the algorithm, then the use of the calculator is more efficient than hand calculation of multidigit problems (National Council of Teachers of Mathematics, 1980). A final consideration is that teachers who have experimented with using calculators in their classroom report a high level of motivation even in the most reluctant learner (Bruni & Silverman, 1976).

Since calculators do appear to offer some advantages for students, how can they be most effectively utilized in the classroom? A number of activities involving the calculator can be employed: practicing basic facts, understanding algorithms, problem solving, and games and puzzles. Using the calculator in these ways can help motivate students, as well as focus their attention on such topics as reasonableness of results. In addition, the fact that calculators do not solve problems, people do, can be reinforced (Carpenter, T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M., Reynolds, R.E., 1981). Some activities for using the calculator in the mathematics classroom can be found in the activity section of the computation strand of this handbook.

Computer-Assisted Instruction

Many school systems have purchased microcomputers with the expectation that this innovation, along with appropriate software, will result in increased student learning. While there are conflicting research reports as to the effectiveness of CAI (computer-assisted instruction) in improving achievement, a synthesis of the available information indicates that the gains in achievement in mathematics when using computers is small, less than .5 standard deviation (Clark, 1985). A factor to consider when using computers is that initially students are very interested in and motivated by using computers (described as the novelty effect). The novelty effect appears to last approximately one year for elementary grade students, but for only two to three months for students in the upper grades. Once the novelty effect is not a factor, gains in achievement were lower (.2 standard deviation).

Some appropriate ways to utilize computers in elementary mathematics classrooms are drill and practice (after a skill has been learned), tutorial programs to let advanced students work ahead of the class, games and simulations that reinforce concepts that have been taught, the use of LOGO for teaching geometric concepts, and simple programming.

Advantages of CAI

- * CAI is an effective way of motivating students in mathematics.
- * CAI permits students to progress at their own pace.
- * Immediate feedback and reinforcement are given.
- * The order of the activities (levels) can be directed.
- * CAI allows for student interaction. It provides a vital situation in which the student can become an "active" learner.

- * CAI can be used as a beneficial supplement to the regular instruction if the teacher possesses a clear sense of how best to integrate CAI or CMI (computer-managed instruction) into the existing curriculum.

Disadvantages of CAI

- * Research does not clearly indicate that achievement is substantially enhanced due to use of CAI.
- * Unless a student has really relearned a concept, practice of incorrect processes can complicate remediation.

Teachers' Attitudes Toward Mathematics

The effective mathematics teacher has a positive attitude about teaching his/her subject. Students can see by the activities in the classroom that the teacher enjoys teaching math (Adams, Ellis & Beeson, 1977). Recently, researchers have begun an interdisciplinary study of teacher's attitudes toward mathematics and student achievement. Resnick (1981) has elaborated on the psychology of the teacher's attitude concerning mathematics education. She analyzed the verbal activity of teachers and concluded that there tends to be a more negative attitude portrayed in mathematics classes, compared to other disciplines. Cooney (1980) has studied how the teacher's belief system about mathematics determines instructional processes and, in turn, how this belief system has been modified by the teaching experience. The teacher's attitude towards mathematics is likely to be an important variable for the student. A positive attitude on the part of the teacher can assist the student in developing a more positive attitude towards mathematics.

Summary

The crucial role of the classroom teacher in effectively promoting mathematics education has been documented by the National Council of Teachers of Mathematics (1980). The extent to which a teacher is knowledgeable about a subject and is able to apply that knowledge is of prime importance in facilitating pupil learning. An effective mathematics teacher demonstrates knowledge and skills in both the content of mathematics and the "how to" of education. Perhaps the most important characteristics of the effective mathematics teacher are an incessant search for new knowledge and better skills, the flexibility to change, and a positive attitude towards mathematics.

**ERRORS:
DIAGNOSIS,
STRATEGIES,
AND
ACTIVITIES**

ERRORS: DIAGNOSIS, STRATEGIES AND ACTIVITIES

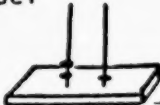
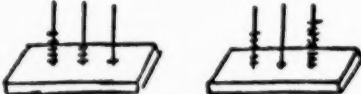
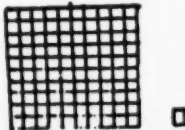
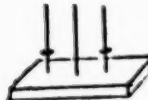
It is vital, of course, that theoretical concepts and constructs have practical applications if they are to make a difference for classroom teachers. The following section of this handbook provides a strong emphasis on practice and pragmatic advice. A major emphasis in this section is remediation: understanding the nature of learning problems a student is experiencing and introducing methods, procedures, and strategies to overcome the problems.

This section builds on the theoretical ideas and concepts about remediation already presented. Within the framework of the six strands of K-5 mathematics instruction (numeration, computation, relations and functions, measurement, geometry, and probability and statistics/grouping), relevant information on errors, diagnosis, strategies, and activities is presented.

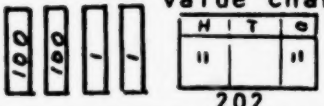
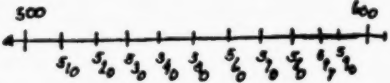
NUMERATION

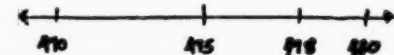
Numeration and the concept of numbers are the bases for the student's beginning understanding of mathematics. Concepts such as one-to-one correspondence, counting, and place value are taught in this strand. Emphasis should be placed on the use of concrete materials to reinforce the student's understanding of numeration. If students fail to fully grasp these concepts, it is unlikely that they will be successful as they progress through the mathematics curriculum.

NUMERATION

Type of Error	Example of Error	Diagnosis	Strategy												
PLACE VALUE Grade K-1	1.  3 Instead of 21	Incorrectly associating place-value model with numerals	Open-ended abacus and other place-value model 												
	2. Counting - 52,53....59,70	Counting error-cannot determine number that comes before or after a given number.	Hundreds board - Take all numerals from board and replace - starting at 1.												
	3. Judges 56 to be larger than 64 Compares ones (6 and 4) Instead of tens (5 and 6)	Digit-comparison errors - cannot make digit comparisons of two numbers to find larger or smaller number	Counting actual sets of objects Place numerals or tally marks on paper-place value chart <table border="1" data-bbox="1450 929 1570 1039"><tr><th>Tens</th><th>Ones</th></tr><tr><td>5</td><td>6</td></tr><tr><td>6</td><td>4</td></tr></table> <table border="1" data-bbox="1648 904 1776 1045"><tr><th>Tens</th><th>Ones</th></tr><tr><td> </td><td> </td></tr><tr><td> </td><td> </td></tr></table>	Tens	Ones	5	6	6	4	Tens	Ones				
	Tens	Ones													
5	6														
6	4														
Tens	Ones														
4. 531,518 - compares ones not tens-and judges 518 to be larger than 531	Ordering error when making digit comparisons-cannot put numerals in order from larger to smaller or from smaller to larger	Place Value Model 													
2-3 50	1. 11 Instead of 101 for 	Omits zero when writing numerals for a given model	Open-ended abacus  51												

NUMERATION

Type of Error	Example of Error	Diagnosis	Strategy
PLACE VALUE Grade 2-3	2. Writes 2002 for "two hundred two"	Incorrect usage of zero in place value	Play money, place value chart 
	3. forty-eight, forty nine, forty-ten	Incorrectly names next numeral in a counting sequence	Hundreds board Remove numerals. Place on board. What would forty-ten look like?
36 4-5	1. Rounds 587 to 510,500 or 5107 when rounding to nearest ten	Rounds numbers inappropriately because of failure to regroup	Place value models or open-ended abacus 
	2. Writes thirty-thousand, fifty-six as 3056 and five hundred seven thousand as 5007000	Incorrectly names and writes place-value positions in numerals Writes numerals for number words incorrectly	Use models and place value charts. Concentrate on place value per se. 15 means 1 ten 5 ones 37 means 3 tens 7 ones Thirty thousand, fifty-six means 30,000 plus 56 = 30,056
	3. Rounds 478 to 470 or 588 to 500 when rounding to nearest ten	Rounds numbers incorrectly	Make non-standard representations for numerals. Use number lines.



NUMERATION

Type of Error

Example of Error

Diagnosis

Strategy

FRACTIONS

Grade

K-3



White section
as $\frac{1}{3}$
Instead of $\frac{1}{4}$

Thinks that 3 is denominator

Misunderstands that the denominator is the number of equal parts

Use many circles and rectangles that are divided into halves, thirds and fourths. Do not shade any parts, but have them marked off, to show 2 halves, 3 thirds and 4 fourths as a whole. Have children write $\frac{2}{2}$, $\frac{3}{3}$, and $\frac{4}{4}$. Then move to shaded parts.

Indicates shaded area as

$\frac{1}{5} \rightarrow$



Doesn't associate meaning with a fraction

Using fraction manipulatives, explain numerator and denominator.

Makes incorrect associations.

Use colored circles to show shaded regions.

0000000 = $\frac{3}{4}$

DECIMALS

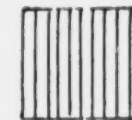
Grade

4-5

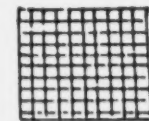
Confuses 0.4, 0.04 and 0.004.

Cannot identify place value positions or doesn't understand $\frac{4}{10}$, $\frac{4}{100}$ or $\frac{4}{1000}$.

Models and transparencies showing decimal fractions and equivalents



Tenths



Hundredths

Cut into strips. 10 hundredths will fit over 1 tenth etc.

$0.59 > 0.6$

Doesn't understand that 0.59 is hundredths and that 0.6 is tenths or 60 hundredths.

Models of tenths and hundredths or use transparencies and overlays. Show equivalency of 6 tenths to 60 hundredths using cut-out strips. Then compare

NUMERATION ACTIVITIES

A. PRENUMBER CONCEPTS

B. NUMBER CONCEPTS

The activities on numeration described on the following pages contain information specific to the skill to be mastered, the materials needed, the procedures to be used, and the suggested grade level for use. Upper level teachers may wish to modify activities designated for lower level students.

A. PRENUMBER CONCEPTS

Skill: ORDERING
Level: K-2
Activity: Full and Empty
Materials: Jars, rice.

Fill several baby food jars or other containers with rice in different amounts, and ask the group to order them from empty to full or vice versa. After arranging the jars, you may want to pour out all the rice and have the children refill the containers. Ask the children to arrange the jars again.

Skill: ORDERING
Level: K-2
Activity: Sequencing Events in Time
Material: Sequencing cards (see below).

You will need a set of sequencing cards depicting events that evolved in time (a tree growing from a seed, the building and melting of a snowman, etc.). Ask the children to look at the cards and decide what happened first, what came next, and so on. Encourage conversation by asking questions: What makes you pick this card? How can you tell that this comes next?

Skill: CLASSIFICATION

Level: K-5

Activity: Intersecting Sets

Materials: Two loops of yarn and attribute blocks.

Position two loops of yarn on a table. From a collection of attribute blocks, begin to form two sets of objects, building each set within one of the circular areas, perhaps "red" and "triangles." As you take each object from the collection, have the children assign it to one of the sets. When they are sure of the procedure, produce a red triangle, and ask them where it should go. They will agree that it is both red and a triangle. Let the children find their own solution to where the block should go. Do not provide a solution at this time; the important concept to be learned is that an object can be a member of more than one set.

SKILL: ORDINAL NUMBERS
Level: K-5
Activity: Using Ordinal Number Words
Materials: Paper and Pencil.

- A. Have the students make a chart of things they did in one day
(sequencing) such as:

First, I got out of bed.

Second, I ate breakfast.

Third, I got dressed.

Fourth, I came to school.

Fifth, _____

Sixth, _____

Seventh, _____

Eighth, _____

- B. Have the students use the newspaper to list the baseball line up:

First batter _____

Second batter _____

Third batter _____

Fourth batter _____

Fifth batter _____

Sixth batter _____

Skill: NUMBER PROPERTIES

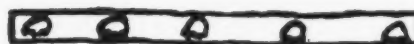
Level: 4-5

Activity: Bottle Cap Math

Materials: Bottle caps.

Directions:

1. Arrange 5 bottle caps in a row on your card like this.



Record this on your paper:

rows columns

$$1 \times 5 = 5$$

(The rows go across and
the columns go up and down).

2. If you turn the card half way around like this:
You still have 5 caps.

$$5 \times 1 = 5$$



Skill: CLASSIFICATION

Level: K-3

Activity: Sort-It-Out

Materials: Different colors, sizes, and shapes of buttons; various coins: pennies, dimes, nickels, and quarters; egg cartons.

The child is asked to sort ordinary objects according to different criteria. The younger child will sort for color, size, or shape, perhaps by putting colored buttons into egg cartons. (The older child will be able to take on more complicated sorting tasks, such as separating items after a shopping trip according to what goes in the refrigerator, what goes in the cupboard, what goes in the cleaning closet, etc.) Putting pennies, dimes, nickels, and quarters into different containers is a game particularly enjoyed by young children.

Skill: CLASSIFICATION

Level: K-5

Activity: Name Another

Materials: None.

Choose a category, like things to eat, things that have wheels, things that go in the water, things that are not found in the house, animals that live in the zoo, transportation, or things that keep us warm. Take turns with children naming things that belong to that category.

Skill: CLASSIFICATION

Level: K-2

Activity: All Sorts of Counters

Materials: Assortment of buttons, blocks, sticks, stones, spools, clothespins, etc.; a number of juice cans or margarine containers.

You need a box or tub with an assortment of buttons, screws, blocks, sticks, stones, spools, clothespins, and so on. You also need a number of juice cans or margarine containers.

Ask the child to sort things that go together and put them in the different containers. Talk about why the child decided to group things the way he/she did. You can also ask the child to count the objects in the different containers.

Skill: NUMERICAL CONCEPTS
Level: K-2
Activity: Conservation of Number
Materials: Vases, flowers, pennies.

With several vases in a row and a pile of flowers nearby, ask the child to get one flower for each vase and then to check his/her work by putting each flower in a vase (thus establishing one-to-one correspondence).

Vary the procedure using pennies in place of vases, telling the child he/she can buy one flower with one penny. Ask him/her to find out how many pennies he/she needs to buy all the flowers.

Skill: ADDITION OF INTEGERS
Level: 3-5
Activity: ADD-IN
Materials: Playing cards, paper, and pencil.

Deal 4 cards to each player, then have each add the cards together. After each round, set cards aside and deal 4 more cards to each player. Score points based on the sum of the integers. This can be varied to apply to subtraction and multiplication. The person with the highest score after all cards are used wins.

Skill: PATTERNING

Level: K-2

Activity: Matching Bead Patterns

Materials: Different shapes and sizes of beads; cards consisting of bead patterns; long shoelaces.

Encourage the children to choose a card and copy the pattern by stringing the beads in correct order.

Skill: PATTERNING

Level: K-2

Activity: What Comes Next?

Materials: Manipulative squares, triangles, circles of differing colors and sizes; pencil and paper.

Draw (or use manipulative shapes) the first four figures of a pattern (square, triangle, square, triangle; or circle, square, circle, square; or small triangle, large triangle, small triangle, large triangle, etc.), and ask the child to finish the line.

SKILL: PATTERNING
Level: K-2
Activity: Patterns With Counters
Materials: 4 x 6 index cards, counters.

You will need several 4 x 6 index cards and any kind of counters you choose. Place the index cards in a linear arrangement, and begin a pattern with the counters by placing one counter on the first card, three on the second, one on the third, three on the fourth, etc. Ask the child to put counters on the next card. Ask: How do you know how many to put there? Why did you use that many counters?

Skill: PATTERNING
Level: K-2
Activity: Matching Patterns
Materials: Wallpaper squares, fabric squares.

Provide several squares of wallpaper, and analyze the pattern (blue stripe, red stripe, blue stripe, red stripe).

Provide two squares of each pattern, and have children find the matching one.

Repeat the activity with fabric scraps. Involve the families by asking each child to bring a patterned fabric or paper scrap from home.

Skill: PATTERNING
Level: K-2
Activity: Involving the Whole Child in Patterns
Materials: Unifix cubes.

Teacher claps hands and touches head; children join in as they recognize the pattern. Stop and start with a new pattern.

Teacher places first few children (one child stands, one child sits, one child stands, one child sits). Additional children join the row and continue the pattern.

Children reproduce a pattern they hear with Unifix Cubes (red, blue, blue, red, blue, blue).

Skill: TRICKY PATTERNS
Level: 3-5
Activity: Pattern 2
Materials: Calculator, paper, pencil.

- | | |
|---|-------------|
| 1. Choose a number | Example: 29 |
| 2. Double it: | 58 |
| 3. Add 4 | 62 |
| 4. Multiply by 5: | 310 |
| 5. Add 12: | 322 |
| 6. Multiply by 10: | 3,220 |
| 7. Subtract 320: | 2,900 |
| 8. Cross out the last two zeros! Compare the answer to the 29 number in Step 1. | |

Why does the pattern work?

Skill: NUMBER CONCEPTS

Level: K-2

Activity: The Hungry Pelican

Materials: 11 3" x 5" index cards or blank pieces white paper; black and yellow felt-tip markers; 11 pieces of white construction paper, 8" x 6"; scissors, pencil, ziploc bag.

- How to make:
1. Trace and cut 11 pelicans out of white paper.
 2. Use a yellow marker to define the beak on both sides.
 3. Use a black marker to add details and to outline the beak heavily on both sides.
 4. Cut 11 3" x 5" cards in half so that you have 22 2 1/2" x 3" cards.
 5. Write numerals 0 through 10 on 11 cards and repeat.
 6. Place the pelicans and numeral cards in a ziploc bag.

- How to Play:
1. Place the pelicans in one pile and the numeral cards in another pile.
 2. Select two numeral cards, and place them in front of a child.
 3. Ask the child, Which numeral is bigger?
 4. After the child has correctly responded, tell the child that the pelican gets very hungry, and he eats the bigger numeral.

The Hungry Pelican (Cont.)

Ask the child to place the pelican next to the biggest numeral so that the beak is open wide toward that numeral.

5. Continue playing until all the cards and pelicans have been used.

Skill: ONE-TO-ONE CORRESPONDENCE

Level: K-5

Activity: Number Bingo

Materials: 8 1/2" x 11" tagboard, counters, dice with dots, die with numerals; contact paper.

Make up to six tagboard cards (8" x 11") divided in fourths. Draw from one to six red circles in each section. Cover with clear contact paper.

Up to six children can play the number bingo at one time. Children take turns to throw the dice. If the child has the number of dots indicated by the die, he/she will cover the dots with counters. The first child to have all the dots covered ends the game. After the children have had experience with numerals, they will find the numeral dice more challenging.

Skill: ONE-TO-ONE CORRESPONDENCE
Level: K-5
Activity: People and Pennies
Materials: Toy people, pennies.

Make a row of eight or nine toy people. Ask a child to give each toy person one penny taken from a container with ten pennies or so. Ask the children if there are more pennies than people. Change the number of pennies or people to obtain different results. In the instances where equivalent sets are obtained, put the pennies back into the container and ask the children: Is there still one penny for each person? Ask one child to prove his/her answer by re-establishing the original one-to-one match.

B. NUMBER CONCEPTS

Skill: NUMBER CONCEPTS

Level: K-5

Activity: Bears and Honey Jars

Materials: Brown construction paper, 10" x 6 1/2" for each bear; white or light tan construction paper, 3" x 3" for each honey pot; felt tip markers, scissors.

- How to Make:
1. Trace and cut out at least 8 bears.
 2. Add eyes, ears, etc., with markers.
 3. Place one numeral on each bear. For example: 3, 4, 5, 6, 7, 8, 9, 10.
 4. Trace and cut honey pots, add detailing.
 5. On each honey pot, place an equation, leaving out one numeral factor. For example: $4 + \underline{\quad} = 9$.
 $5 + \underline{\quad} = 10$. $3 + \underline{\quad} = 7$.

- How to Play:
1. Place the bears in a line in front of the child.
 2. Put the problem cards in a pile to the left of the child, face up.
 3. Have the child pick up a problem card and decide which bear has the missing numeral. The child can then place that card below the bear and pick another card.
 4. Check his/her work when he/she is done. Remember to praise child's efforts, as well as successes.

Bears and Honey Jars (Cont.)

- Variations:
1. This game can be used for addition, subtraction, and multiplication by changing the problems on the honey pots.
 2. In the beginning of second grade, have the problems going only to 10. By the second half of the year, you can increase the amounts up to and including 18.

Skill: COUNTING
Level: K-2
Activity: Counting on With Unifix Cubes
Materials: Unifix cubes.

Have the children count out a certain number of cubes and hide them under one hand. Give the children a few more cubes and ask them to verbalize the number hidden and count on to the total number of cubes. Repeat several times. Keep the total small.

Skill: COUNTING
Level: K-2
Activity: Guess How Many?
Materials: Cubes.

Hide a few cubes in one hand and a few more in the other. Ask the children to guess how many you have in each hand. Keep the total constant for a while. Children will catch on to the idea of regrouping. Repeat with a different total.

Skill: COUNTING
Level: K-2
Activity: Matching with Cards
Materials: Playing cards.

Lay out playing cards, one through ten, on a table or floor. Place the rest of the cards (after removing the face cards) next to the ones spread out. One or more children taking turns will try to match the top card from the deck with one of the displayed cards. Encourage counting of the spots on the cards and the reading of the corresponding numeral.

Skill: COUNTING
Level: K-2
Activity: Clothespin Counting
Materials: Cardboard, clothespins.

Make five squares (later you may want to make five more) with zero to four dots on them. Use heavy cardboard for the cards. You will need a box of clothespins. Ask the child to pin the corresponding number of clothespins on each card.

Skill: COUNTING
Level: K-4
Activity: Number Patterns
Materials: Toothpicks, construction paper.

Ask the children to make as many different arrangements as possible with two toothpicks. Talk about the arrangements. Repeat with three, then four toothpicks. As the children become experienced with the procedure, you may want to have them glue the toothpicks on construction paper. Repeat the activity at another time, using other materials such as buttons, dot stickers, clothespins.

Skill: COUNTING
Level: K-2
Activity: The Concept of Zero
Materials: Empty film containers, pennies, objects such as beans, rice cotton, salt.

Collect several film containers. Place pennies in every container except one, and close them. Try this with beans, rice, cotton, salt. Have the children add items singularly by counting.

Skill: NUMERALS

Level: K-2

Activity: Writing in Sand or Cornmeal

Materials: Salt or cornmeal; box lid or cake pan.

Pour enough salt or cornmeal in a box lid or a cake pan to cover the bottom. The children trace numerals in the salt or cornmeal. Shake lightly to erase and start over. Repeat this activity often.

Skill: NUMERALS

Level: K-2

Activity: Dominoes With Numerals

Materials: Dominoes (can be made from cardboard).

Make or use a set of dominoes with numerals on one end and dots on the other. Tell the children that dots must be matched to corresponding numerals and numerals to the corresponding dots. Play the game as usual.

Skill: TRICKY PATTERNS
Level: 3-5
Activity: Pattern 1
Materials: Calculator, paper, pencil.

Example

- | | |
|-------------------------------------|----------------|
| 1. Choose any two digit number: | 18 |
| 2. Reverse the digits: | 81 |
| 3. Add your number and its reverse: | $18 + 81 = 99$ |
| 4. Write the divisors of the sum: | 11, 9 |

Try the pattern again with another number.

Why does the pattern work? Here's a way to start thinking about it. A two-digit number can be represented as $10t + u$ (ten times the tens digit plus the units digit). What happens when you add a number and its reverse?

Skill: PLACE VALUE
Level: 3-5
Activity: Round-Off Raceway
Materials: Tagboard to make a "raceway," two margarine bowls, tagboard cut into squares.

Construct a "raceway" of your choice on a tagboard using tens (10, 20, 30,...). For each "stop," use two margarine cups. One of the margarine cups has a different-colored square piece that tell how many ones you have, the other, how many tens. Draw a square from each cup. Now the child rounds off the number to the nearest ten. They then move to that spot on the round-off raceway if it's ahead of them. If it's behind them, they can't move on that turn. The winner is the one who reaches the finish line first.

Skill: PLACE VALUE
Level: 4-5
Activity: Math in the Newspaper
Materials: Tagboard, newspaper, markers.

To be able to read and fully comprehend articles in a newspaper, one needs understanding of a surprising amount of mathematics. Have the children read newspaper articles and tabulate numbers found onto place value columns (made from tagboard). Also, ask the children to convert word recognition of numbers into digits.

Skill: PLACE VALUE
Level: 3-5
Activity: Human Place Value Chart
Materials: Tagboard for numerals, ball.

Teacher calls out a decimal number. The students who have any part of that numeral come to the front of the room and place themselves in the proper position so the number will read correctly. A ball indicates the decimal point. Example: If the teacher calls the number 19 hundredths, the students holding 1 tenth and 9 hundredths would position themselves next to the ball decimal point.

Skill: EXPANDED NOTATION
Level: 4-5
Activity: Expand It
Materials: Playing cards, paper, and pencil.

Face cards and tens are valued at zero, and aces at one. First player tells dealer how many cards to deal. Cards are laid face up as they are received, and the player writes the expanded notation of the number represented by the cards. If a student correctly writes the number in expanded form, points are earned for each card used. Example, 4 cards dealt in this order: King, nine, ace, two. The number is 0912...

Skill: PLACE VALUE
Level: K-5
Activity: The Store
Materials: Objects to be used in "store", pennies, dimes, tagboard, paper, and pencil.

The children decide on two things to "buy" and write down their names and prices. They then put the money needed on their place-value board, followed by computing their totals and comparing.

This activity can be extended to include estimation and multiplication skills.

Skill: NUMERATION
Level: 4-5
Activity: Place Value Quick Fingers
Materials: Calculator, paper, markers, hundreds and tens chart.

Children enter digits on a calculator until the display is full. Ask children to read the display as each new digit is added. For example:

Teacher says: "One."

The children push 1

Teacher says: "Two."

The children push 2.

They read the display "Twelve."

"Now enter 3."

The students enter 3.

Students name the new number, "One hundred twenty-three."

This activity is excellent for enrichment purposes in reading place value and increasing the students' awareness of place value concepts.

Variation: This activity could also be used in conjunction with a tens and hundreds chart. The students shade in ten and they have a "tens" digit on the display, etc.

Skill: NUMERATION
Level: 3-5
Activity: Off to the Races With Skip Counting
Materials: Calculators.

Many calculators have a constant addend feature. With this capability, students can use the calculator to count.

To count forward by ones, enter

$$1 + 1 = \underline{\quad} + 1 = \underline{\quad} + 1 = \dots$$

As each number is displayed, ask students to name it aloud. You may try a race. The only rule for the race is to stop exactly on 100. This encourages students to carefully observe the display, emphasizing many place value concepts, and increases their awareness of patterns.

Based on how long it takes to race to 100, have students guess how long it will take to count to 200, to 1000.

"Suppose we start at 900 and go to 1000. How long will it take?" This is your keying sequence for this

$$1 + 900 = \underline{\quad} + 900 = \underline{\quad} \dots$$

Try skip counting by two. Enter

$$2 + 2 = \underline{\quad} + 2 = \underline{\quad} + 2 = \underline{\quad} + 2 \dots$$

Name each number as it displays. Try the same with 5's and 10's.

You may go backward (subtract) from 100 by entering 1 -- 100 ==...

Skill: PROBLEM SOLVING
Level: K-2
Activity: Logic Problems for Preschoolers
Materials: Attribute blocks.

Show one attribute block at a time, arranging them in a row. First, show the big blocks. "Let's find out what is in this package. Tell me what each object is as I take it out." Arrange the blocks in a row. Show the small circular blocks in the same color order as the large ones, arranging them under the large block of matching color. Have the children identify the first small circle. "What do you think the next block will be? What makes you think so?" Repeat with blocks of other shapes. "Tell me when you think the package will be empty," and so on.

Skill: PROBLEM SOLVING
Level: K-2
Activity: Logic Problems for Preschoolers
Materials: Square tiles.

Ask the children to arrange four tiles in as many different ways as they can think of. Have children count their own arrangements.

Repeat the activity asking children to use six tiles.

Skill: PROBLEM SOLVING
Level: K-2
Activity: Take One, Take Two
Materials: Objects.

Place ten objects in front of two players. Each player may take one or two objects at her turn. Players alternate. The person who takes the last object wins.

Ask the children how they can be sure of winning. Would it be better to start first or second.?

Try the game with more or fewer objects. Can the children see a pattern?

COMPUTATION

Computation is generally emphasized in the elementary school mathematics curriculum. Computation involves using addition, subtraction, multiplication, and division. Addition and multiplication involve "putting together" while subtraction and division involve "taking apart." Students must understand the meaning of these operations and be able to choose the correct operation in order to solve problems in everyday life.

EXAMINING PROCEDURAL AND CONCEPTUAL ERRORS

One should not conclude that misunderstood/absent concepts are responsible for every computational error. It is possible that the child's difficulty is with the algorithm (procedure) or merely inattention to the operation sign. However, when conceptually based errors are treated as if they are procedurally based, the specific erroneous behavior(s) may disappear, only to reappear in other related computations. It is important that, where possible, conclusions be made as to which specific concept(s) or principle(s) the child does not understand. Enright, in his Diagnostic Inventory of Basic Skills, provides us with examples of seven error clusters.

1. REGROUPING - This cluster of errors shows that the student has little understanding of place value or the arithmetic steps to show it.

$$\begin{array}{r} 28 \\ + 8 \\ \hline 216 \end{array} \quad \begin{array}{r} 13 \\ - 7 \\ \hline 16 \end{array} \quad \begin{array}{r} 46 \\ \times 2 \\ \hline 812 \end{array} \quad \begin{array}{r} 11 \\ 7 \overline{)98} \end{array}$$

2. PROCESS SUBSTITUTION - In this error cluster, the student changes the process of one or more of the computation steps and creates a different algorithm that results in an incorrect answer.

$$\begin{array}{r} 42 \\ + 3 \\ \hline 75 \end{array} \quad \begin{array}{r} 45 \\ - 2 \\ \hline 7 \end{array} \quad \begin{array}{r} 34 \\ \times 2 \\ \hline 14 \end{array} \quad \begin{array}{r} 8 \\ 2 \overline{)6} \end{array}$$

3. OMISSION - This cluster of errors is indicated when a student leaves out a step in the process or leaves out part of the answer. An omission error differs from a process

$$\begin{array}{r} 4.75 \\ +.62 \\ \hline .37 \end{array} \quad \begin{array}{r} 45 \\ - 2 \\ \hline 3 \end{array} \quad \begin{array}{r} 5.61 \\ \times .43 \\ \hline 83 \\ 44 \\ \hline 523 \end{array} \quad \begin{array}{r} 1 \\ 4 \overline{)7} \\ \hline 4 \end{array}$$

substitution error by representing an incomplete, rather than a different, algorithm.

4. DIRECTIONAL - In this error cluster, the computation is correct, but the steps are performed in the wrong direction and/or order.

$$\begin{array}{r} 42 \\ + 3 \\ \hline 9 \end{array} \quad \begin{array}{r} .55 \\ - .3 \\ \hline .52 \end{array} \quad \begin{array}{r} 12 \\ \times 23 \\ \hline 24 \\ 36 \\ \hline 384 \end{array} \quad \frac{2}{3} \times 3 = \frac{2}{9}$$

5. PLACEMENT - These errors are often computed accurately, but because the numbers are written in the wrong place, the answers will be incorrect.

$$\begin{array}{r} .43 \\ + .26 \\ \hline 6.9 \end{array} \quad \begin{array}{r} 45 \\ - 2 \\ \hline 34 \end{array} \quad \begin{array}{r} 12 \\ \times 23 \\ \hline 36 \\ 24 \\ \hline 60 \end{array} \quad \begin{array}{r} 40 \\ 5 \overline{)20} \end{array}$$

6. ATTENTION TO SIGN - By ignoring the sign, the student performs the wrong operation. This generally occurs when the student uses the "shape" of the item as his or her sole clue to which operation to perform.

$$\begin{array}{r} 1 \\ + 2 \\ \hline 2 \end{array} \quad \begin{array}{r} 5 \\ - 2 \\ \hline 7 \end{array} \quad \begin{array}{r} 6 \\ \times 4 \\ \hline 10 \end{array} \quad \frac{4}{6} \div \frac{2}{6} = \frac{2}{6}$$

7. GUESSING - In this cluster, the errors often lack logical quality, indicating a lack of basic understanding of the processes or skills to be used.

$$\begin{array}{r} 1 \\ + 2 \\ \hline 21 \end{array} \quad \begin{array}{r} 4 \\ - 4 \\ \hline 4 \end{array} \quad \begin{array}{r} 6 \\ \times 4 \\ \hline 46 \end{array} \quad \begin{array}{r} 6 \\ 2 \overline{)6} \end{array}$$

PRACTICE IN ERROR ANALYSIS

Listed below are some examples of the types of errors a child might make if numeration concepts were not understood. These examples were taken from Helping Children Understand and Use Numerals by Jon M. Engelhardt, Robert B. Ashlock and James H. Wiebe.

Examine each error pattern for the specific concept(s) that may be misunderstood or absent. Space is provided for you to note your conclusions. The authors' conclusions follow. Remember, do not focus exclusively on what the child did, but rather on deciding what misunderstood numeration concept or principle would lead the child to respond in that particular way.

$$\begin{array}{r} 74 \\ +56 \\ \hline 1210 \end{array}$$

$$\begin{array}{r} 35 \\ +92 \\ \hline 127 \end{array}$$

$$\begin{array}{r} 18 \\ +67 \\ \hline 715 \end{array}$$

$$\begin{array}{r} 56 \\ +97 \\ \hline 1418 \end{array}$$

Your conclusions:

Authors' conclusions: Although one might be tempted to simply conclude that the child's difficulty is with the regrouping process, there appear to be two misunderstood numeration notions: "only one digit is written in a position or place" and "each place in a multidigit numeral has a single value ten times greater than the place to the right and one-tenth the place to its left (place relationship and regrouping)." Since pairs of digits are squeezed into single spaces, the child seems to be aware that the pair of digits is describing a single place value. However, the child appears unable to rename the numeral with one digit in a place or is unaware that such a representation is necessary in our system.

Both potentially misunderstood notions are based on the idea of group, that is, "when representing how many, objects are grouped into sets of a specified size into sets of sets." It is the relationship among the group sizes that leads to the meaning of regrouping--the formation of a new group size when one reaches ten of a given group size--and hence the presence of only one digit in a place. Thus, in addition to the two previously mentioned notions, the even more basic idea of grouping may not be understood.

$$\begin{array}{r} 30 \\ +6 \\ \hline 9 \end{array}$$

$$\begin{array}{r} 21 \\ +5 \\ \hline 8 \end{array}$$

$$\begin{array}{r} 30 \\ +14 \\ \hline 8 \end{array}$$

$$\begin{array}{r} 42 \\ +3 \\ \hline 9 \end{array}$$

ERROR IDENTIFICATION PRACTICE

Look at the examples below and identify the error cluster, using Enright's classification system:

- | | | |
|----------------|-------------------------|----------------------|
| 1. Regrouping | 2. Process Substitution | 3. Omission |
| 4. Directional | 5. Placement | 6. Attention to Sign |
| 7. Guessing | | |

1. _____	$\begin{array}{r} 3 \\ +2 \\ \hline 1 \end{array}$	$\begin{array}{r} 5 \\ -2 \\ \hline 10 \end{array}$	$\begin{array}{r} 6 \\ \times 4 \\ \hline 2 \end{array}$	$\frac{1}{4} + \frac{2}{4} = \frac{2}{16}$
2. _____	$\begin{array}{r} 48 \\ +21 \\ \hline 9 \end{array}$	$\begin{array}{r} 433 \\ -122 \\ \hline 3 \end{array}$	$\begin{array}{r} 3 \\ \times 5 \\ \hline 5 \end{array}$	$\begin{array}{r} 1 \\ 36 \overline{) 47} \\ \underline{36} \end{array}$
3. _____	$\begin{array}{r} 9 \\ +6 \\ \hline 51 \end{array}$	$\begin{array}{r} 45 \\ -2 \\ \hline 34 \end{array}$	$\begin{array}{r} 84 \\ \times 63 \\ \hline 252 \\ 504 \\ \hline 3024 \end{array}$	$\begin{array}{r} 31 \\ 4 \overline{) 7} \\ \underline{4} \\ 3 \end{array}$
4. _____	$\begin{array}{r} 50 \\ +37 \\ \hline 510 \end{array}$	$\begin{array}{r} 48 \\ -42 \\ \hline 24 \end{array}$	$\begin{array}{r} 34 \\ \times 2 \\ \hline 86 \end{array}$	$\begin{array}{r} 12 \\ 4 \overline{) 168} \end{array}$
5. _____	$\begin{array}{r} 4 \\ +3 \\ \hline 34 \end{array}$	$\begin{array}{r} 13 \\ -7 \\ \hline 10 \end{array}$	$\begin{array}{r} 4 \\ \times 8 \\ \hline 84 \end{array}$	$\begin{array}{r} 8 \\ 5 \overline{) 80} \end{array}$
6. _____	$\begin{array}{r} 28 \\ +8 \\ \hline 106 \end{array}$	$\begin{array}{r} 13 \\ -7 \\ \hline 14 \end{array}$	$\begin{array}{r} 123 \\ \times 3 \\ \hline 129 \end{array}$	$\begin{array}{r} 1200 \\ 4 \overline{) 800} \end{array}$
7. _____	$\begin{array}{r} 28 \\ +8 \\ \hline 26 \end{array}$	$\begin{array}{r} 43 \\ +12 \\ \hline 211 \end{array}$	$\begin{array}{r} 46 \\ \times 2 \\ \hline 82 \end{array}$	$\begin{array}{r} 10 \\ 6 \overline{) 93} \end{array}$

COMPUTATION

Type of Error	Example of Error	Diagnosis	Strategy
SUBTRACTION Grade 2-5	$\begin{array}{r} 7 \\ 384 \\ - 52 \\ \hline 3212 \end{array}$	Regrouping when not necessary	Reteach place value using concrete models. Use objects to subtract $8 - 5 = 3$. Show that regrouping is not necessary.
	$\begin{array}{r} 6 \\ 702 \\ - 469 \\ \hline 243 \end{array}$	Difficulty with multiple regrouping; difficulty regrouping with zero.	Oral practice renaming numbers such as 702 = 7 hundreds 2 ones; 70 tens 2 ones; 69 tens, 12 ones before working with place value models. May have to use simpler problems, then move to more complex.
MULTIPLICATION Grade 3-5	$\begin{array}{r} 4 \\ 62 \\ \times 7 \\ \hline 461 \end{array}$	Reversal of digits in regrouping	Reteach regrouping using concrete objects; use place value chart to correct transposing. Show that $2 \times 7 = 1$ ten and 4 ones and discussion of keeping "likes" together.
	$\begin{array}{r} 602 \\ \times 407 \\ \hline 434 \\ 248 \\ \hline 2914 \end{array}$	Ignores zero during computation	Review concept of zero procedures. Make sure child learns to justify the recording of each digit and each place should be accounted for. Could have to resort to semi-concrete or concrete level for reteaching using manipulatives.
	$\begin{array}{r} 87 \\ \times 45 \\ \hline 435 \\ 348 \\ \hline 8415 \end{array}$	Misaligned product—due to lack of understanding of place value	Review place value with place value model and chart. Use boxes for vertical alignment.

Type of Error	Example of Error	COMPUTATION Diagnosis	Strategy
ADDITION Grade 2-5	$\begin{array}{r} 942 \\ 2346 \\ + 408 \\ 17 \\ \hline 12191 \end{array}$	Poor alignment of digits in the column arrangement	Use lined paper sideways so lines form a guide to vertical columns; draw boxes for vertical alignment
	$\begin{array}{r} 298 \\ 37 \\ \hline 2+9+8+3+7 \end{array}$	Adds each digit separately disregarding columns	Abacus; Start with manipulatives for teaching place value; use materials that show tens and ones. Demonstrate with these materials.
SUBTRACTION Grade K-1	$\begin{array}{r} 9 \\ - 4 \\ \hline 6 \end{array}$	Inadequate knowledge of facts and relations	Count real objects or use flash cards. Practice mental computations
	$\begin{array}{r} 45 \\ - 8 \\ \hline 43 \end{array}$	Assumes that commutative property of addition holds for subtraction; regrouping may not be understood	Place-value models, open-ended abacus, play money Make a construction using visual aids. Construct a situation to show you can't remove 8 objects from 5 objects.
2-5	$\begin{array}{r} 57 \\ - 32 \\ \hline 89 \end{array}$	Disregards the operation sign	Trace the sign; count objects; demonstrate addition as putting together and subtraction as taking apart.
	$\begin{array}{r} 538 \\ - 192 \\ \hline 466 \end{array}$	Failure to regroup; Subtracting the smaller digit from the larger regardless of where it appears	Place-value models; charts teach children to associate models with numerals.

COMPUTATION

Type of Error	Example of Error	Diagnosis	Strategy
DIVISION Grade 4-5	$\begin{array}{r} 422 \\ 2 \overline{)804} \end{array}$	Difficulty in using zero in the dividend	Use concrete manipulatives to show why $0 \div N = 0$. Practice using zero as a place holder.
	$\begin{array}{r} 28 \\ 6 \overline{)1248} \end{array}$	Omits zeros in the quotient	Make a record of each place value position indicated in the dividend. Have child justify each step. Use boxes to show place value positions. $\begin{array}{r} \\ 6 \overline{)1248} \end{array}$
	$\begin{array}{r} 41 \\ 70 \overline{)287} \end{array}$	Ignores zero in a divisor	Perform computations with divisors that are multiples of 1, 10 and 100. Review the distributive property. Have children compare the results of their computations.
	$\begin{array}{r} 1106 \\ 4 \overline{)5624} \end{array}$	Difficulty regrouping when finding the quotient or does not use remaining values after subtracting	Demonstrate regrouping of smaller dividends with concrete models
	$\begin{array}{r} 8 \\ 5 \overline{)4030} \\ 40 \\ \hline 0 \end{array}$	Does not complete computations when dividing after subtracting in the body of the computation when a 0 is obtained.	Stress place value. Expand the numbers. Ask following series of questions: If I had 4030 pennies, could I give 1000 to each of 5 children? (No) Could I give 900 to each of 5 children? (No) 800? (Yes). How many would I have left if I gave 800 to each of the 5 children? (30). Could I give 100 to each child? (No) 10 to each child? (No) 9, 8, 7, 6 - yes. How many are left? (0)

Type of Error

Example of Error

Diagnosis

Strategy

(continued)

Therefore $800 + 6 = 806$ and
 $4030 \div 5 = 806$

Also, use mnemonic device
 to show procedure. "Does
 (divide) McDonalds (multiply)
 Sell (subtract) Canned
 (compare to insure remainder
 is smaller than divisor)
 Biscuits (bring down next
 digit)"

79

4-5

$2/3 = \square/6$ May choose
 2 for the $\square$
 since $6 \div 3 = 2$

1. Difficulty using the
 Equivalent - fraction -
 rule. Does not use
 appropriate method to
 find equivalent fractions

Fraction strips cut from
 construction paper in halves,
 fourths, eights, thirds,
 sixths. Place various
 strips to show equivalencies.

FRACTIONS

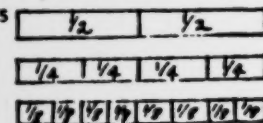
Grade

4-5

$6/8 = 2/4$

2. Doesn't simplify properly
 Computes $6 \div 3 = 2$ and then
 $8 \div 2$. Does not understand
 equivalencies.

Fraction strips and the use
 of patterns



$.17 + 16.2 = 17.9$

Does not observe place
 value principles involving
 decimals

Discussion of adding things
 that are alike - apples plus
 apples, cats plus cats, dollars
 plus dollars. This applies to
 decimal parts; tenths plus
 tenths, ones plus ones,
 hundredths plus hundredths, etc.

Type of Error	Example of Error	COMPUTATION Diagnosis	Strategy
DECIMALS Grade 4-5	$\begin{array}{r} 43.75 \\ + 4.21 \\ \hline .4796 \end{array}$ or $\begin{array}{r} 36.4 \\ 2.53 \\ \hline 3.893 \end{array}$	Counts number of decimal places in both addends and counts off that many in sum. Confuses rule that applies for multiplication.	Reteach the meaning of decimal numerals and other related fractional ideas using fraction strips and tenths and hundredths chart.
	$\begin{array}{r} 5.2 \\ \times 0.4 \\ \hline 20.8 \end{array}$	Computes correctly but does not understand the rule for multiplying fractions.	Teach that a tenth of a tenth is a hundredth by using drawings or transparencies.
	$0.31\overline{)62}^2 \text{ or } 0.05\overline{)0.25}^{\cdot 5}$	No conception of why or how the decimal works.	Reteach the meaning of decimals using drawings or transparencies (overlays).

COMPUTATION ACTIVITIES

The activities on computation described on the following pages contain information specific to the skill to be mastered, the materials needed, the procedures to used, and the suggested grade level for use. Upper level teachers may wish to modify activities designated for lower level students.

SKILL: ADDITION
Level: K-2
Activity: Addition With Cuisenaire Rods
Materials: Cuisenaire rods.

Ask the children to find single rods that are equal, the same length as a train of two or more rods. Children must find the black rod to match the train of green and purple, and so on. Allow enough time for children to explore and discover relationships.

Skill: ADDITION AND SUBTRACTION
Level: K-2
Activity: Bead Stringing
Materials: Box of beads; several shoelaces.

Ask the children to string a number of red beads on their shoelace and then, for example, add a number of green beads. Ask the children to count how many beads they have all together. Use different combinations. Then ask them to remove a number of beads, and, again, ask them to count the beads to see how many they have left.

SKILL: NUMBER PROPERTIES

Level: K-5

Activity: Finding the Special Properties of 1 and 0

Materials: Dominoes.

Directions:

1. Stack all the dominoes face down.
2. Then take turns drawing a domino and saying the sum of the dots.
3. Example: If you draw a 6, you say 2 dots + 4 dots = 6 dots.
4. Ask: What can you say is always true when one side of the domino is blank?

SKILL: COMPUTATION

Level: 3-5

Activity: Is It Odd or Even?

Materials: Poster board, markers, paper, pencil.

Directions to students:

1. Count the number of people in your class. Is this number odd or even?

Record your information like this:

Number of People ____ Odd or Even? ____

2. Ask some of your classmates what size shoe they wear. How many students wear an odd number size? How many wear an even size? Tally this information on a chart like this.

Shoe	Size
Odd	Even

3. Ask some of your classmates to tell you the number of people in their family. How many families have an odd number of members? How many families have an even number of members? Make a chart.

Skill: ADDITION
Level: 1-4
Activity: New Names
Materials: Tag Board and markers.

Have the students work in small groups. Instruct each group to design a poster that illustrates at least ten names for a given number. Each group should work with a theme of its choice.

Skill: SUBTRACTION
Level: 3-5
Activity: Subtraction Whiz
Materials: Two sets of number cards marked 0-18, paper, and pencil.

Have the students write the "difference" of all subtraction facts from 0-9 on a piece of paper. Then have each player select 19 cards. After the first player circles a difference, the other player lays down as many pairs of cards as possible showing two numbers with that difference. Have the students take turns circling differences for each other. The winner is the player who runs out of cards first.

Skill: SUBTRACTION
Level: K-5
Activity: Subtraction Wheels
Materials: Construction paper, markers.

Have the students draw subtraction wheels. Explain that number at the hub of the wheel must be the difference between the two numbers at the rim that are connected with the same spoke. Have the students complete the wheel. Then have them create their own wheels with different numbers on the hub.

Skill: SUBTRACTION
Level: K-5
Activity: Subtraction Boxes
Materials: Construction paper, shoe box, markers.

Teacher constructs a shoe box with an "in" and out" square cut from the top. Children construct a strip of 10 happy faces and 10 separate ones. Teacher directs children to represent $12 - 6$ by showing 12. (Be sure the children exchange a ten strip for 10 ones.) Then the children remove 6 pieces. For manipulative purposes, use the cut-out shoebox in exchanging tens for ones.

Skill: ADDITION AND SUBTRACTION
Level: K-2
Activity: The Shark
Materials: Fish crackers, blue construction paper.

You will need fish crackers and blue oceans (a sheet of blue construction paper for each child).

Place from zero to eight crackers in each ocean. Have each child tell you how many fish he/she needs or will need to put back so each starts with six. Tell the story of Jaws, the Shark, first requiring regrouping of fish. "Three swim to the top. Now how many are at the bottom? Now the shark swims to the top. How many are at the top/bottom/whole sea? Now the shark swims by and gobbles up one of the fish at the bottom. How many are left?" Children eat one and answer: five. The shark goes away, and two more fish come along Mix regrouping, addition, and subtraction.

Skill: ADDITION AND SUBTRACTION
Level: K-3
Activity: Adding Integers: Subtraction Integers
Materials: Counters.

Give the child unequal sets of counters such as eight and fourteen and ask him/her to make them equal, or give the child a set of twelve counters and ask him/her to divide them into two equal sets.

Skill: NUMERALS
Level: K-2
Activity: Egg Carton Game
Materials: Egg cartons, marbles, magic markers.

Number the egg carton cups from zero to eleven (with young children, you may want to explore only the first three or four numerals).

Ask the children to put a marble in their egg cartons, close the top and shake. The children then place the egg carton on the table, open the top, and identify the numeral in the cup in which the marble has fallen.

Skill: ADDITION AND SUBTRACTION
Level: K-2
Activity: The Shark
Materials: Fish crackers, blue construction paper.

You will need fish crackers and blue oceans (a sheet of blue construction paper for each child).

Place from zero to eight crackers in each ocean. Have each child tell you how many fish he/she needs or will need to put back so each starts with six. Tell the story of Jaws, the Shark, first requiring regrouping of fish. "Three swim to the top. Now how many are at the bottom? Now the shark swims to the top. How many are at the top/bottom/whole sea? Now the shark swims by and gobbles up one of the fish at the bottom. How many are left?" Children eat one and answer: five. The shark goes away, and two more fish come along Mix regrouping, addition, and subtraction.

Skill: ADDITION AND SUBTRACTION
Level: K-3
Activity: Adding Integers: Subtraction Integers
Materials: Counters.

Give the child unequal sets of counters such as eight and fourteen and ask him/her to make them equal, or give the child a set of twelve counters and ask him/her to divide them into two equal sets.

Skill: NUMERALS
Level: K-2
Activity: Egg Carton Game
Materials: Egg cartons, marbles, magic markers.

Number the egg carton cups from zero to eleven (with young children, you may want to explore only the first three or four numerals).

Ask the children to put a marble in their egg cartons, close the top and shake. The children then place the egg carton on the table, open the top, and identify the numeral in the cup in which the marble has fallen.

Skill: MULTIPLICATION

Level: K-2

Activity: Match the Cards

Materials: Simple problem-solving cards, beans, margarine tubs, paper, pencils.

Using simple multiplication problem cards, have the children solve each problem by using the beans to count out into representative groups, placing beans in margarine tubs. Then have the children draw pictures to show their groups.

Skill: MONEY/MULTIPLICATION

Level: K-5

Activity: Money Magic

Materials: Place-value board, old math books, Unifix cubes, paste, paper, pencil.

The children paste pictures of nickels cut from old math books in an increasing pattern on their number lines. Have the children count how much each row is worth in cents and record the numbers. If the children have difficulty adding the sums, have them place five Unifix cubes over each nickel and then regroup the cubes on their place-value board.

Skill: MULTIPLICATION

Level: 4-5

Activity: Multiplication Tic-Tac-Toe

Materials: A deck of cards with multiplication riddles, two different sets of markers, and tic-tac-toe game labeled with number sentence.

Construct a deck of cards containing multiplication riddles, such as:

6 bugs in a jar

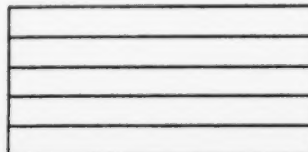
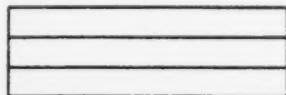
6 jars

How many bugs in all?

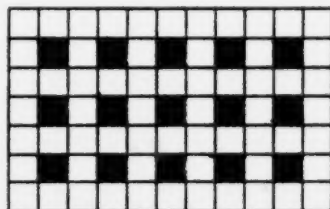
Construct a tic-tac-toe game board labeled with multiplication sentences, such as $6 \times 6 = 36$. Children draw from deck to match riddle with sentence. If a match occurs, marker is placed over that space on the game board. The winner is the first player to capture a row, column, or main diagonal.

Skill: MULTIPLICATION
Level: 4-5
Activity: Modeling Multiplication
Materials: Markers, paper, scissors.

Have the student chose two numbers to be multiplied such as 5 x 3. For this example they would cut 5 slots in one paper and 3 slots in the other.



Next they put one page over the other on their desk, so that it looks like this



Ask: "Do you see why this diagram shows that $5 \times 3 = 15$? and $3 \times 5 = 15$?"

Continue with other examples.

Skill: DECIMALS

Level: 4-5

Activity: Identifying Tenths, Hundredths, Etc.

Materials: Graphing paper, marker, decimal cards made by teacher ($2/10$, $1/100$, etc.).

Using graphing or grid paper, have children box in 10 squares followed by 100 squares. Use a fraction to explain shaded regions. Ask students to shade in decimal representation. This is an excellent activity to demonstrate the equivalency of fractions and decimals.

Skills: COMPUTATION
Level: 3-5
Activity: A Game For Two Players
Materials: 3 x 5 index cards, material for game board, markers.

Here's a game designed to provide experience in mental computation and practice with mathematical vocabulary for students in upper elementary school. The directions can be adjusted for students in lower grades. Take a stack of 3" by 5" index cards and on each card write one direction such as the following: "Move to the next multiple of 4," "move to the next multiple of 5," "...of 6," "....10," "...12", "Multiply by 2," "...by 3," "...4," "...5," "...6"; "Divide by 1," "...by 2," "...3" "...4, "...5," "...10," "...12," "...20": "Square your number"; "Divide by the number you're under"; and "Reverse the digits of your number."

Make a game board with sixty squares and number them consecutively from 1 to 60 for each reference. A rectangular grid of squares or even a spiral or other pattern is fine. Window shades or the back of vinyl wallpaper make excellent board material, but ditto paper or oak tag is fine. To play, shuffle the deck and place the cards in a pile, face down. Place a playing piece in square 1 for each player, and decide who goes first. Draw a card and move as indicated; do not move if you are told to go to a square already occupied or a square not on the board. The winner is the first person to get to 60, or the person who lands on the highest number after a predetermined time limit is reached.

Skill: COMPUTATION

Level: 3-5

Activity: Prime Number Throw

Materials: Materials to make a hundreds board, dice, bottle caps.

Make a chart like a hundreds board. Children work in groups of 2. Instruct them as follow:

1. Throw 2 or 3 dice.
2. Add or Multiply the numerals on the dice together.
3. If the result is a prime number, place a bottle cap on that number on the master chart.
4. Game is over when all prime numbers have been covered.

Winner--player with the most prime numbers covered.

Skill: COMPUTATION

Level: 3-5

Activity: Mental Methods

Materials: Hundreds charts, markers, overhead projector, transparency.

Good mental methods require the understanding of place value and number properties. This strategy promotes the understanding of the reasoning behind mental calculation before students attempt mental shortcuts.

Using a hundreds chart can help develop a number of mental strategies such as:

- (1) Make an overhead of hundreds chart.
- (2) Children use markers on their chart to solve $\begin{array}{r} 53 \\ +36 \\ \hline \end{array}$ mentally.
- (3) Children transfer work with hundreds chart to computing mentally.

(excerpt)
(from)
(hundreds)
(chart)

30



(53)
63
73
(83)

84 85 86 87 88 (89)

→ + 6 →

$$\begin{array}{r} 53 \\ +30 \\ \hline 6 \\ 89 \end{array}$$

Skill: PROBLEM SOLVING
Level: K-5
Activity: Writing in Math Classes
Materials: Paper and pencil.

If students can write clearly about mathematical concepts, it is apparent that they understand them. For this reason, good writing is a part of the problem solving process.

- A. Have the children rewrite story problems they don't understand.
- B. Have the students generate story problems of their own.
- C. Rewrite a section of the math textbook narrative they do not understand.

Skill: PROBLEM SOLVING
Level: K-5
Activity: Amazing Mazes
Materials: Teachers can design their own mazes and then XEROX them so that children have their own copy.

The child must find his way through a maze. Simple mazes follow a path with no pitfalls. The next stage involves mazes with only one trick. As the child gets more experienced, mazes of increasing complexity are used.

Skills: FRACTIONS
Level: K-5
Activity: Fractions With Pattern Blocks
Materials: Pattern Blocks.

Many fractional relationships are discovered by manipulating the Pattern Blocks. If the parallelogram has the value of one, then the triangle is one-half, because two triangles make one parallelogram. Three triangles make a trapezoid. Three parallelograms make a hexagon. Two trapezoids make a hexagon.

Skill: FRACTIONS
Level: K-5
Activity: APPLES
Materials: Fruit (apples, bananas), paper plates, knife.

Have apples for snack. Sit down with a small group of children. Cut one apple into two equal pieces called "halves." Put the two pieces together to form a whole apple. Cut the apple into four pieces called "fourths." Cut another apple into "thirds." Again, put the apple back together to show the children the "whole" apple.

If you serve bananas, let one child cut the banana and another child choose the part.

Skill: DECIMALS
Level: 3-5
Activity: In Between
Materials: A deck of 30 cards labeled with decimal numbers.

Two, three, or four children can play. Each player draws two cards and announces a decimal between the two cards he is holding. If the player is correct, he or she keeps the cards; otherwise, the next player tries to name an appropriate number. If the next player names an appropriate number, then he or she can keep the cards as well as taking his or her regular turn. Otherwise, the player loses a turn, and the cards are passed on to the next player. Play continues accordingly. A pair of cards may be passed around once. If no player can name an appropriate number, the cards are shuffled into the deck. The winner is the player with the most cards.

Skill: COMPUTATION
Level: 2-5
Activity: Calculator Calculating
Materials: Calculator, paper, pencil, problem cards.

Ask students to solve a few simple problems with the calculator. Go slowly, and be sure all students arrive at the correct solution so you can be certain they understand the operation of the calculator. Allow students to confront a situation that may be special problems. For example:

$$\begin{array}{r} 18 \\ 14 \\ \hline +9 \end{array}$$

Although visually only one "+" is seen, students using calculators must enter the addition symbol twice: $18 + 14 + 9 =$

Ask students to solve more complex problems from problem cards. You might try a race to see who wins (the first child to reach the solution).

Skills: ESTIMATION
Level: 3-5
Activity: Round It
Materials: Dice, deck of playing cards, pencil, paper.

First player rolls dice. The larger number is the number of cards dealt, and the smaller number is the number of places from the right to round off (ex. 1 = ones place, 2 = tens place, etc.). If a student rounds correctly, the rounded number is the number of points earned.

Skill: ESTIMATION
Level: 3-5
Activity: Label Lingo
Materials: Assorted labels from various consumer products, pencil, paper.

Using assorted labels, children utilize nutritional information for problem solving. After a teacher-directed lesson, the children will be prepared to generate their own problems using labels.

Extend activity to include estimation skills, such as estimation of ingredient's proportions. Stress that ingredients are listed in order from the largest proportion to the smallest.

Skill: DIVISION

Level: 3-5

Activity: Division Raceway Game

Materials: Large lima beans labeled with the numbers from 1 to 50; a game board with a track labeled with the numbers from 1 to 9; dice, margarine tub.

Using the gameboard, players (2-4) put playing pieces on start. Each player:

1. Rolls dice and moves indicated number of places.
2. Picks a bean from the margarine tub without looking.
3. Finds the greatest multiple of the number shown on the landing spot contained in the number shown on the bean. If correct, player stays on new landing spot.
4. Winner is first player to land on "Finish".

Example: Player lands on 5, Picks the bean labeled 49. States that the greatest number of 5 in 49 is 9--that is, $(9 \times 5) + 4 = 49$.

Skill: FRACTIONS
Level: 3-5
Activity: Fraction Strips
Materials: Colored posterboard.

Fraction concepts expressed in terms of fraction strips challenge children to attend to the meaning of the fractions involved. Cut strips from different colors of posterboard to represent each of the various fractions. The shortest strip is square 2 cm by 2 cm. The next strips are longer: 4, 6, 8 cm in length.

Below are listed several possible classroom activities.

- Activity 1: Have children manipulate strips showing the relationship of a whole to $1/2$, $1/3$, $1/4$, etc.
- Activity 2: Have children repeat Activity 1 and also add written symbols that apply. Incorporate adding and subtracting.
- Activity 3: Have the children compare two or more fractions. The child should discuss and make comparisons verbally before finding the actual fraction strip and making physical comparisons.
- Activity 4: For more advanced students, have the children compare fractions with like denominators, but different numerators.
- Activity 5: In this activity, a basic for concepts of equivalent fractions is begun. Select one strip as the whole, and have the students find all the possible ways to match this strip by laying out strips of one color end-to-end. Then discuss the names of fractions represented by the different strips.

Skill: PROBLEM SOLVING
Level: 2-5
Activity: Make a Drawing and Work Backward
Materials: Worksheets, transparency, overhead projector, pencils.

These two problem-solving skills, making a drawing and working backward, enhance the student's understanding of this concept. Distribute copies of problems to children. Use a transparency to explain to the children how each clue in the problem can be drawn. Stress the usefulness of drawing their solution. Following completion of the drawing, point out that the solution in their drawing can be examined before they consider the operation to be used. Point out that this vertical thinking can help them decide on the feasibility of their answer.

Skill: SOLVING
Level: K-5
Activity: Problem-Solving Detectives
Materials: Pencil and paper.

A detective story setting seems to capture the interest of older students. Have the students construct stories on their own using a detective story theme, and draw illustration to elaborate.

Skill: APPLICATION OF NUMERICAL CONCEPTS TO OTHER SUBJECTS
Level: 4-5
Activity: Mathematics Term Paper
Materials: Pencil and paper.

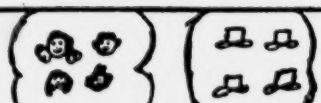
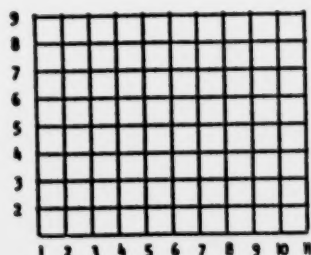
The interdisciplinary nature of this activity encourages students to see how mathematics is vitally connected with different facets of life and how it relates to job and interest. Children in late elementary, middle school, and high school courses are assigned a math term paper, which requires the student to illustrate the use of mathematics in relation to another discipline or to show a practical job-related application of mathematics.

Since the paper should not be a rehashing of a standard topic in math or a book report of a life and/or studies of a known mathematician, there is no single set of resources to which a student can turn. Therefore, directions should be given on narrow topics and how they are influenced by mathematics.

RELATIONS AND FUNCTIONS

The strand of relations and functions encompasses concepts such as comparisons of quantities and ordered pairs. Understanding relations and functions provides students with tools necessary for organized thinking.

Many concrete experiences provide opportunities for students to develop these concepts. Students should not only be given opportunity to work with greater-than and less-than relations, but should be exposed to the relationships of as-many-as, more-than, and fewer-than. These classification activities develop the bases for the understanding of ordered sets.

Type of Error	Example of Error	Diagnosis	Strategy										
One-to-One Correspondence Grade K-1	 Are there more hats or more heads? Child responds: hats (or heads)	Doesn't understand one-to-one correspondence	Use actual objects for experiences in the classroom. Have children distribute one pair of scissors per child, one book per child, etc. Prepare a sheet for children to match one nose per face, etc.										
2-3		Begins graph at one instead of zero	Class or small group discussion of zero as a beginning point. Discuss launching of a rocket or satellite and the count down to zero. Explain that before you have one of anything you have none or zero. Discuss age of a baby before its first birthday. Children then draw their own graphs beginning at zero.										
Greater than/ Less than Grade 2-3	$8 < 12$ Child reads as "8 is greater than 12"	Doesn't understand greater than and less than	Bulletin board with 2 sets of flannel or poster board objects with arrows indicating one-to-one correspondence. Have children count the two sets and decide which has the greater number.										
Type Patterns Grade 4-5	<table><tr><th>In</th><th>Out</th></tr><tr><td>5</td><td>7</td></tr><tr><td>9</td><td>11</td></tr><tr><td>13</td><td>17</td></tr><tr><td>10</td><td>12</td></tr></table> Child fills in 14, 16, 18	In	Out	5	7	9	11	13	17	10	12	Cannot determine patterns from given examples	Prepare actual "input" "output" boxes. Have 2 students work together to put "in" and take "out" combinations involving all 4 operations. Start with simple numbers and operations and gradually move to complex numbers
In	Out												
5	7												
9	11												
13	17												
10	12												

RELATIONS AND FUNCTIONS

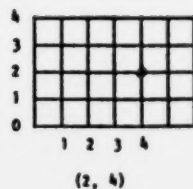
Type of Error

Example of Error

Diagnosis

Strategy

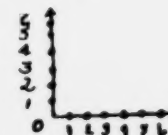
4-5



Incorrectly locates ordered pairs

1. Make a large graph on the floor from plastic or paper. Have children walk to various points on the graph and stand on the point designated.

2. Place the following on an overhead



Point out the "X" axis and "Y" axis

RELATIONS AND FUNCTIONS ACTIVITIES

The activities on relations and functions described on the following pages contain information specific to the skill to be mastered, the materials, the procedures to be used, and the suggested grade level for use. Upper level may wish to modify activities designated for lower level students.

Skill: FUNCTIONS

Level: K-5

Activity: Ordered Pair Role-Playing

Materials: Masking tape, cards with numbers.

Using masking tape, draw a large grid on the classroom floor. Give students cards with numbers. Talk about ordered pairs. Have children walk on the "grid" and act out the ordered pairs.

Skill: FUNCTIONS

Level: 3-5

Activity: Get the Message

Materials: Grid paper, markers, ordered pair cards.

Have the students draw a rectangular coordinate system with an X-axis and a Y-axis. Then do these things to get the message.

1. Find the ordered pairs below on the grid.

(3,10)	(4,6)	(7,6)	(8,4)	(11,2)
(2,10)	(4,8)	(7,4)	(8,2)	(10,2)
(2,9)	(5,6)	(9,2)	(10,1)	
(3,9)	(6,8)		(11,1)	
(3,8)	(6,6)		(11,1)	
(2,8)			(10,1)	
			(10,0)	
			(10,0)	
			(11,0)	

2. Connect the points in the order they are given.
3. Do not connect the points from one column to points from another column.

Skill: LESS-THAN RELATION

Level: K-3

Activity: Which Has the Least?

Materials: Various objects, drawing paper, markers, crayons.

Use manipulative objects (9 of one object, 8 of another object, and 6 of another object) to have students decide which set is more and less. Ask students questions so they can manipulate objects in answering.

Have the children draw nine happy faces, eight trucks, and six sailboats.

Ask students to use crayons to draw line to match happy faces to trucks. Ask students to explain "Are there as many faces as trucks? Are there as many faces as sailboats? Which is the least? How do you know?"

Skill: MORE-THAN-RELATION

Level: K-3

Activity: Hanging-Up Numbers

Materials: Cardboard, felt pockets, markers, yarn for ties, paste.

Draw a cardboard shirt with ties at the neck and felt pockets with numerals.

Draw cards picturing objects with different numbers.

The students place cards in the pockets that are more than the number indicates.

Then talk about matching the correct picture to the number.

Skill: AS-MANY-AS RELATION

Level: K-3

Activity: Which Group Has the Same?

Materials: Set cards (pictures, paste), envelopes, markers.

To make the set cards, paste matching pictures onto colored file cards. To make the envelopes, paste or draw happy faces onto each envelope. The envelopes should be big enough to hold three set cards. Make three set cards and one envelope for each of the numbers 1 through 10.

Mix cards with different numbers of objects. Envelopes should match cards.

Ask students to put the card that has "as many as" inside the envelope that is appropriately marked.

Skill: RELATIONS AND FUNCTIONS

Level: K-5

Activity: Graphing Relations and Equalities

Materials: Different colored buttons, grid (markers and paper for graph boards).

Construct graph boards. Along the bottom and right hand side, mark off number lines so that the bottom left corner is the zero point for both lines.

Fill in vertical and horizontal lines from the number points to complete a 20 x 20 grid.

The children understand that the horizontal lines are "main streets" with their number names at the bottom, and vertical lines are "side streets" with their number names on the left.

Directions: Each street crossing is named by the number pair that tells the name of the two streets which cross.

Put red buttons on all crossings that add up to 7.

Put blue buttons on all crossings that add up to 3.

Put white buttons on crossings that add up to 15.

Skill: RELATIONS AND FUNCTIONS

Level: 3-5

Activity: Showing Relations and Equalities as Mappings on the Number Line

Materials: Constructed number lines, paper arrows that are mobile.

Directions: You will need two number lines that are alike and arrows.

Place the number lines one under the other so that their numbers match up.

Put arrows from the bottom line to the top to connect the numbers. Do you recognize a pattern? (they are alike)

Now match these pairs.

7 10 2 13

5 8 0 11

Can you recognize the relation?

$$\boxed{} \longrightarrow \boxed{} + 2$$

Skill: RELATIONS AND FUNCTIONS
Level: 2-5
Activity: Order Two or More Numbers: "Ranko"
Materials: 1/2 egg carton per player, 25 index cards (for 4 players)

Make 25 index cards numbered 1-25 on one side. Write the word "RANKO" on the back. With a knife, cut slightly between spaces on the egg carton, so cards can stand in the carton.

Directions: The dealer shuffles the RANKO cards. Then he gives each player a card. Each player places his card in the slot nearest him. Each player puts 5 cards in his egg carton.

The extra cards are placed face down in a pile. To take turns, each picks up another card, puts it into a slot, and discards on old card.

To win, 5 cards must be in order. The winner calls out "RANKO."

Sample winning hands;

5, 6, 9, 12, 25

1, 4, 10, 11, 24

2, 6, 13, 19, 21

Skill: RELATIONS AND FUNCTIONS
Level: K-3
Activity: Matching Related Items
Materials: Paper, crayons, pencils.

Directions: Draw two circles on your paper. Write the color words (blue, brown, green) in the first circle. In the other circle, write the names of 3 friends. Check to see what color eyes they have. Draw an arrow from the eye color to the name (match friends to their favorite color on the back of your paper).

Skill: RELATIONS AND FUNCTIONS

Level: K-3

Activity: Sorting Buttons

Materials: Various colored buttons, straws of different lengths.

Directions: Use your buttons of different color and sizes.

1. Put them into different piles by colors. How many piles do you have?
2. Put them into piles by size.

Variation: Compare length of different-sized straws.

Skill: RELATIONS AND FUNCTIONS

Level: K-3

Activity: Attribute Blocks

Materials: Attribute blocks (yellow, red, and blue construction paper, scissors).

Using attribute shapes (blocks), ask children to find those that are alike.

If attribute blocks are not available, make the following set of 12 per student:

1 yellow square

1 red square

1 blue square

1 yellow rectangle

1 red rectangle

1 blue rectangle

1 yellow triangle

1 red triangle

1 blue rectangle

1 yellow circle

1 red circle

1 blue circle

Determine how many different combinations can be alike.

GEOMETRY

Geometry is the strand of mathematics concerned with plane and solid figures. Concepts of geometry are presented throughout the elementary school curriculum, beginning with recognition of shapes and patterning in the early grades to the calculation of perimeters and areas in the upper grades.

GEOMETRY

Type of Error	Example of Error	Diagnosis	Strategy
Grade 2-3	 Indicates that the triangles are congruent	Does not understand that congruent means both same shape and same size	Have children hold two pieces of construction paper together and cut a shape from the two pieces of paper. Have children put the 2 shapes together to see that they are the same size and same shape.
4-5		Confuses rays with line segments	<u>Acting Out & Demonstrate</u> Use a straw to demonstrate a line segment. To demonstrate a ray, place a ball of yarn in a box. Have a child walk away while pulling the end of the skein through a hole in the box. Imagine that the child walks, & walks and walks to demonstrate a ray.
4-5	Child writes A = diameter B = radius 	Confuses radius and diameter	Use a marker to show diameter and radius on an apple. Then 'cut' the radius and diameter from the apple. Children then practice marking diameter and radius on circles they cut from construction paper.
4-5	 Student chooses these three figures as <u>all</u> displaying <u>right</u>	Does not understand difference between right, obtuse, and acute angles.	Use a geoboard with rubber bands to show relationship between angles. Then transfer to paper.

GEOMETRY

Type of Error

Example of Error

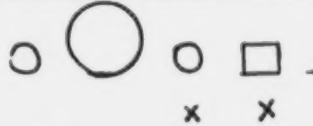
Diagnosis

Strategy

Grade

K-1

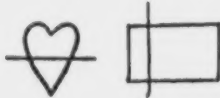
Child chooses figures with X below as being congruent.



Does not understand that congruent means both same size and same shape

Discussion of things that are the same size and shape. Follow discussion with comparison of large and small balls and large and small blocks or cubes. Measure using string or yarn.

122



Marks as showing symmetrical parts

Misunderstands meaning of symmetrical parts

Cut many shapes that are symmetrical from construction paper. Have children fold the paper shapes in half to show symmetry. Then have children cut shapes to show symmetry. Children can then mark lines at symmetry on prepared sheets.

GEOMETRY ACTIVITIES

The activities on geometry described on the following pages contain information specific to the skill to be mastered, the materials needed, the procedures to be used, and the suggested grade level for use. Upper level teachers may wish to modify activities designated for lower level students.

Skill: GEOMETRY
Level: K-3
Activity: Squares
Materials: Unifix cubes, geoboards.

Children take turns making horizontal or vertical line segments of any length on their geoboards. A child who completes a square with no nails in the middle places the Unifix cubes within the square to claim it. The children continue playing until all squares are claimed.

Skill: THREE-DIMENSIONAL SOLIDS
Level: K-3
Activity: Matching Balls, Boxes, and Cans
Materials: A ball, an orange, an onion, several types of boxes, several different cans (or objects similar to these).

Examine the different items. Analyze their surfaces, edges, and corners. Discuss flat and round surfaces. Classify the shapes. Explore if they roll or not, and so on.

Skill: PROBLEM SOLVING
Level: K-3
Activity: Shape and Color Matrix
Materials: White tagboard, markers, contact paper

Make a matrix board out of a 12" x 12" piece of white tagboard by dividing it into sixteen squares. Make blobs of yellow, blue, and red on the vertical axis of the card. On the horizontal axis, draw the outlines of a triangle, circle, and square. Cover the card with contact paper. Use the appropriate shapes from the attribute blocks set.

Demonstrate to the children how to select a shape and then place it on the appropriate matrix, matching both color and shape. Then encourage the children to work on it as a group.

Skill: GEOMETRY/MEASUREMENT
Level: K-5
Activity: Tangrams
Materials: Tagboard, dot paper, tracing paper, pencils.

Using dot paper, cut and label polygons in a set consisting of five triangles, a square, and a rhombus (a type of parallelogram). These polygons are called Tangram Pieces.

Children use the tangrams to do the following:

- a. Make congruent shapes.
- b. Make a quadrilateral that does not have right angles.
Use all pieces.
- c. Use three pieces to make a square.
- d. Use two pieces to make a pentagon.
- e. Make some designs that look like real objects. Such designs are called tangrams. Trace the designs.

Skill: GEOMETRY
Level: K-5
Activity: Toothpick Shapes
Materials: Toothpicks, construction paper, pencils.

Directions:

Use toothpicks to make the specified figures on construction paper. Then sketch around the results.

- a. Two quadrilaterals that are not similar.
- b. A triangle having congruent angles. Can you use 3 picks to make the triangle? 6 picks? 9 picks? What can you say about the sides of each triangle?
- c. Two nonrectangular parallelograms that have different shapes. Use the same number of picks for each. How must the two parallelograms be different?
- d. A parallelogram with exactly one right angle. Can you do it?
- e. A pair of rectangles that are similar but not congruent;
a pair of triangles that are similar but not congruent.
- f. Regular polygons with 3 sides, 4 sides, 5 sides, 6 sides, 8 sides.

Skill: TWO-DIMENSIONAL SHAPES

Level: K-5

Activity: Shape Bingo

Materials: Tagboard, spinner, crayons or markers, margarine tub.

Make several Bingo-type cards divided in nine areas. Trace and color appropriately the shapes from the Pattern Block set. Make a spinner showing the same shapes.

For his/her turn, the child spins and if he/she finds a figure on his/her card like the one indicated by the spinner, he/she takes such a shape from the tub and covers the appropriate space on her card. The winner must have three objects in a row or a full card. You may wish to let the children decide how the winner will be decided. You may prefer not to have a winner and play until all children fill their cards.

Skill: TOPOLOGICAL SPACE

Level: K-5

Activity: The Pipe Cleaners Game

Materials: Pipe cleaners.

Have the children use pipe cleaners to form a variety of closed and not-closed curves. Ask each child to sort his/her shapes into two piles.

MEASUREMENT

Measurement is the process of assigning numbers to the physical qualities of length, area, volume, weight (mass), or temperature. Time is also measured, but it lacks a physical quality. Each of the other mentioned physical qualities can be measured by means of one or more units.

The process for determining most measures of length and capacity are direct or made by applying the appropriate unit directly to the object being measured. Weight (mass), temperature, and time cannot be measured in the direct manner. The characteristics of properties, weight, temperature, and time require an instrument that measures by indirectly translating the measurable property into numbers.

Children's understanding of measure begins as they engage in exploration and conserving with nonstandard units. Activities with nonstandard units bridge the gap between exploratory work (readiness activities) and the introduction of standard units.

The sequence of measurement strategies (exploratory work, introduction of nonstandard units, and finally work with standard units), will provide experiences that will help the children develop motor skills and learn to discriminate on the basis of size, shape, and color. The exploratory and nonstandard units develop prerequisite skills for understanding other concepts.

MEASUREMENT

Type of Error

Example of Error

Diagnosis

Strategy

Grade

K-1

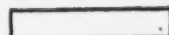


3 o'clock 5 o'clock 12 o'clock

Does not read both hands on a clock

Manipulatives

Prepare a clock with 2 different color hands (minute hand-black; hour hand--red). Make sure children understand that they must read both hands.



5 cm

Misreads rulers

Use simple rulers that are marked in only one unit (cm ruler with no mm's indicated). Provide many experiences where children measure real things from their environment.

130

2-3

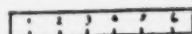
Measurement as repeated placement of a unit



Lines up the end of the object to be measured with "1" on the ruler

Manipulative Materials

Use nonstandard units to measure. Make rulers by pasting units on a strip of cardboard or posterboard.

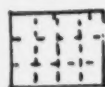


Child estimates that car is 10 cm long.

Estimates too large or too small in linear measure

Pose questions that need solutions. "Will this table fit under the window?" "Do we have enough paper to cover the table?" Use nonstandard measuring tools before allowing children to work with standard measures.

4



4-5

Calculates

Area = 14 units

or

Perimeter = 12 sq. units

Cannot distinguish between area and perimeter

Acting Out

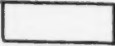
Use masking tape to mark a rectangle on the floor. Have children to stand "inside" the rectangle in marked off squares to "cover" the rectangle.

Count the number of children in the rectangle to understand

159

160

MEASUREMENT

Type of Error	Example of Error	Diagnosis	Strategy
(continued)			Children line up "around" the rectangle. Ask "how many children does it take to go around the rectangle?" to identify the perimeter
4-5	4 2  2 4 Perimeter = $4+2+4+2$	Assumes that you must add 4 sides to find the perimeter of all 4 sided figures instead of $2s + 2s = P$	Have children make a chart with the length of all sides of squares & rectangles listed. Then encourage children to look for a pattern.

MEASUREMENT ACTIVITIES

The activities on measurement described on the following pages contain information specific to the skill to be mastered, the materials needed, the procedures to be used, and the suggested grade level for use. Upper level teachers may wish to modify activities designated for lower level students.

Skill: MEASUREMENT

Level: K-3

Activity: How Many Nails?

Materials: A balance scale, several medium-sized nails, several small but dissimilar objects such as a toy car, a cork, a block, a ping-pong ball.

Ask the children to balance the scale by placing an object in one pan and the appropriate number of nails in the other. Record the result: "car = five nails." After the children have had some experience with this activity, ask them to estimate the weight of the object in nails. Repeat the activity some time later with additional objects.

Skill: MEASUREMENT

Level: K-3

Activity: How Many Cupfuls?

Materials: Milk cartons (pint, quart, half-gallon), measuring cups, dry and/or wet pourable materials.

Open the top of the milk cartons. Ask the child to guess how many cupfuls will fill the container.

Let him/her check by filling the container with that number of cups. If the cup is smaller (half-cup measure, for example), will it take more or less cupfuls? Reverse the procedure, starting with full containers and pouring the contents out in the measuring cups.

Skill: MEASUREMENT

Level: K-3

Activity: Ready to Measure

Materials: Box of toothpicks, clothespins, paper clips, or whatever material you may wish to use in measuring activities with non-standard units.

Ask the children to measure different objects by placing toothpicks end to end. Measure a book, a pencil, a toy. Measure again with a different unit. Why are the results different? Ask the children to estimate first and then measure. Encourage good guesses.

Skill: MEASUREMENT

Level: K-3

Activity: The Giant and the Elf

Materials: Several cutouts of large feet and several of small feet.

Tell the children the story: "In Never-Never Land, there lives a giant who has a funny way of measuring his land. He goes heel-to-toe along. In the same land there also lives an elf, who always does just the same thing as the giant. When he tried to heel-and-toe the giant's garden, he got an answer different from the giant's. Why?" Let the children measure a "giant's garden" using both sets of cutouts. "Suppose John's shoe is used instead. What number do you think we might get?"

Skill: MEASUREMENT

Level: K-5

Activity: Which Is Heavier?

Materials: A collection of objects whose weight and size seem disproportionate, such as washers, ping-pong balls, golf balls, balsa wood, hardwood, cotton, etc.

Ask the children to guess by looking at two objects which one is heavier. Let them hold the objects and guess again. Later, they will use the balance and find out which one is heavier.

Take a walk around the classroom and identify two objects. Let the children lift them, one at a time, and decide which is heavier (the red truck or the fire engine, the cardboard block or the wooden block, the elephant or the giraffe). Encourage the children to describe how it feels to lift a heavy object, a light object, etc.

PROBABILITY AND STATISTICS

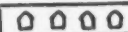
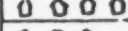
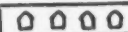
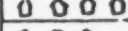
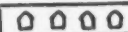
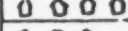
Probability is the likelihood that an event may occur, whereas statistics is the process of gathering and analyzing data.

This strand includes the skills of organizing data into tables, charts, and graphs. "Data fluency" ensures that the students can transfer information into visual form. Students can also learn to draw inferences from the data they have collected and tabulated. Graphs used may include bar graphs, line graphs, circle graphs, and pictographs. Tables and charts enable the student to organize data so that specific information can be located quickly.

In teaching probability, it is essential that the students understand the relationship of "event" to its "occurrence" or "outcome." The effective mathematics teacher should introduce the terms (event, impossible events, likely event, certain event, and equally likely event) to upper elementary students by means of carefully sequenced activities.

In choosing activities, linking practice in determining probabilities with problems of current interest to the students can provide strong motivation for delving into the concepts of probability and statistics.

PROBABILITY AND STATISTICS

Type of Error	Example of Error	Diagnosis	Strategy												
Grade K-1	<table><tr><td>1986</td><td></td></tr><tr><td>1985</td><td></td></tr><tr><td>1984</td><td></td></tr><tr><td>1983</td><td></td></tr></table> Each  represents 5  's. Child says that there are 4 houses represented on the graph for 1986, 2 for 1985, 4 for 1984 and 3 for 1983.	1986		1985		1984		1983		Counts number of pictures instead of number - represented by each picture on a picto-graph	Have children count ten pennies and exchange them for one dime. Record 10 pennies = 1 dime. Explain to children that we often use 1 object to represent more than one. Explain that often there is not enough room to record all items counted and that we often use 1 to represent many.				
1986															
1985															
1984															
1983															
2-5	<table><tr><th colspan="2">VACATIONS</th></tr><tr><td>Beach</td><td>5</td></tr><tr><td>Mountains</td><td>4</td></tr><tr><td>Grandparents</td><td>5</td></tr><tr><td>Disneyworld</td><td>9</td></tr><tr><td>Other</td><td>3</td></tr></table> How many people went on vacation to the beach and to their grandparents? Child responds 5.	VACATIONS		Beach	5	Mountains	4	Grandparents	5	Disneyworld	9	Other	3	Doesn't understand how to read and interpret a chart.	Ask questions such as -- how many brothers and sisters do you have, how many buttons are on your shirt, in what month is your birthday, etc. and have children stand in a circle drawn on the floor that puts children in the correct category. Then on a large chart have children record the correct numbers. Ask questions that have children learn to make interpretations from the chart questions such as the how many in all, fewer, more, or together.
VACATIONS															
Beach	5														
Mountains	4														
Grandparents	5														
Disneyworld	9														
Other	3														

PROBABILITY AND STATISTICS

Type of Error	Example of Error	Diagnosis	Strategy
Grade 4-5	Jim had 10 red marbles and 10 black marbles in a bag. Without looking in the bag, what are his chances of drawing a red marble from the bag in one try? Response: 1 in 20	Adds total number of marbles and thinks if you draw 1 your chances of drawing a red marble would be 1 in 20. The student forgets that there are 10 red marbles.	Make a model or actually place marbles or representatives in a bag and draw a marble out. Ask questions as children draw out marbles (one at a time). Continue concrete activities until children can predict which color marble will be drawn and until children "figure" out a way to scientifically predict the odds of choosing one color, etc.
139	The game that Lisa and Bob are playing uses a spinner like the one shown in the drawing. What are the chance of Lisa spinning a 7?  Response: 7	Misunderstanding or no conception of probability.	Discussion of how many numbers are shown on the spinner and how many 7's are on the spinner. Use spinner and record to determine odds.
	What are the chance of Bob spinning an even number? Response: 4	Partial understanding of probability	Provide first hand experiences where child actually spins a spinner and records results to see that there are 4 even numbers out of a total of 8 4 in 8 or 1 in 2.

PROBABILITY

Type of Error

Example of Error

Diagnosis

Strategy

Grade
4-5

Jonathan made the following scores on his six weekly spelling tests.

What was his average score?

SPELLING TEST SCORES	
Week	Score
1	25
2	18
3	23
4	19
5	24
6	23

Solution -
 $25+18+23+19+24+23 = 132$

$$6 \times 132 = 792$$

Added scores and multiplied by number of scores. Doesn't understand the meaning of an average (or mean)

Begin by using only two scores with the children. Let children learn to set up tables of their own and explain the meaning of averages, (or mean). Use the newspaper to get tables or charts for students to generate averages.

PROBABILITY AND STATISTICS ACTIVITIES

The activities on probability and statistics described on the following pages contain information specific to the skill to be mastered, the materials needed, the procedures to be used, and suggested grade level for use. Upper level teachers may wish to modify activities designated for lower level students.

Skill: PROBABILITY AND STATISTICS
Level: 3-5
Activity: Rolling a Number Cube
Materials: Number cubes.

Have the children roll their number cubes to solve the first two problems (see next page). Remind them that for problems 3-10, that the denominator for this experiment is always 6, because 6 is the total number of possible outcomes, as they will discover in problem 2.

Tell the children that probability is often written in the form of a fraction. To simplify the concept of this activity the wording

_____ out of _____

is also used, and fractions are not reduced to lowest terms.

QUESTIONS

1. Roll the number cube. List the possible ways the cube can land.
2. How many ways can the number cube land?
3. What is the probability that you roll a 4? ____ out of 6, or $1/6$.
4. What is the probability that you roll an odd number?
____ out of 6, or $3/6$.
5. What is the probability that you roll a number less than 5 ____ out of 6
or ____/6.
6. What is the probability that you roll a fraction? ____ out of 6, ____/6.

Skill: BAR GRAPHS

Level: K-5

Activity: Chip Bar Graph

Materials: Colored counting chips, crayons, paper bag, copies of graph.

Have the students look at the graph below. Discuss the labels on the graph. Have each child pick 12 chips from a paper bag. Show them how to arrange the chips on the graph according to color. Point out that each child may have a different answer.

Ask the children to use their crayons to color the graph squares to match the chips they picked.

Ask questions to ensure that children can read the bar graph, such as:

1. How many green chips did you pick?
2. How many yellow and red chips did you pick in all?
3. What color counting chip did you pick the least?
4. What color counting chip did you pick the most?

Color

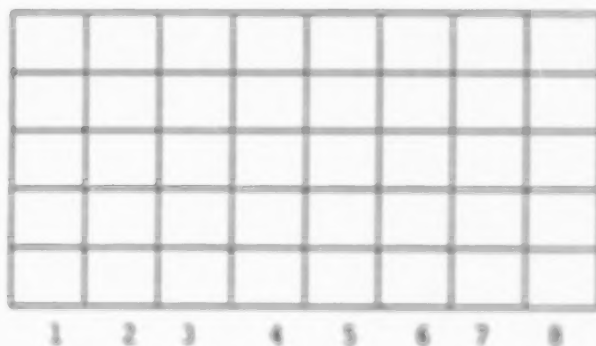
Yellow

Pink

Green

Red

Purple



Skill: PROBABILITY AND STATISTICS

Level: K-5

Activity: Roll the Cubes

Materials: Number cubes, pencils, crayons, copies of tally chart, rulers, and paper.

Demonstrate how to record data using tally marks. Have the child roll a pair of number cubes 36 times, indicating the sum of the cubes by marking a tally mark in the appropriate box on the table (such as below). Then have them count their tallies and record the total for each sum in the row labeled "total."

Help the students to use their data to construct a bar graph. For each sum, the children must find their total on the vertical axis and draw a bar extending to that point.

Directions: Roll a pair of number cubes 36 times. Add the numbers that face up. Show the results in the table below. Also, make a bar graph.

	2	3	4	5	6	7	8	9	10	11	12
Tally											
Total											

Skill: PROBABILITY AND STATISTICS

Level: 3-5

Activity: Flip and Roll

Materials: Pennies, number cubes, possibility charts, pencils.

Children learn to use manipulatives in working with probability and record keeping at an early age by using the activities suggested below:

1. Have the students flip a penny.

How many ways can the penny land?

List the ways the penny can land.

2. Have the students call a number cube.

How many ways can the number cube land?

List the ways that the number cube can land:

Show the students how to keep a record of their cube tosses by using tallies.

Tell them to make one tally mark for each time they toss the cube.

3. Have the students flip a penny and then toss

a number cube. Have the students list all

the possible ways that this can happen. They can

complete the following table to show the possibilities.

Flip and Roll (Cont.)

H = Heads

T = Tails

H	H	H	H	H	H
1	2	3	4	5	6
T	T	T	T	T	T
1	2	3	4	5	6

4. Relate to multiplication: $2 \times 6 = \underline{12}$

Skill: PROBABILITY AND STATISTICS

Level: 3-5

Activity: Double Spin

Materials: Number spinner, copies of table, pencils.

Have children spin the pointer twice and record the sum obtained until they have completed the table. Then instruct them to use the table to solve the problems.

Directions: Spin the pointer twice. Find the sum of the two spins. Write your answer in the table.

Use this table to solve these problems.

1. How many ways can you get a sum of 6? Name them.
2. How many ways can you get a sum that is an odd number?
3. How many ways can you get a sum that is a multiple of
3?
6?

First Spin

Second Spin	1	2	3	4	5
1					
2					
3					
4					
5					

Total

**MENTAL
ARITHMETIC
AND
COMPUTATIONAL
ESTIMATION**

MENTAL ARITHMETIC AND COMPUTATIONAL ESTIMATION

In An Agenda For Action (1980), it was recommended that the teaching of mental computation and estimation be integrated into the mathematics curriculum. Until recently, the primary estimation skill that was taught was rounding. Its relationship to estimation was never clearly established in students' minds (Driscoll, 1981). Emphasis has been placed on the development of estimation skills due to the fact that technology (computers and calculators) has made heavy demands for using estimation skills to determine if results obtained are reasonable. A second reason for the present emphasis on estimation is because approximately 80 percent of all mathematics performed by adults involves the use of estimation skills, yet many adults are unable to estimate well.

Definitions of Mental Computation and Computational Estimation

Mental computation and computational estimation have similarities and differences. Similarities are that both are performed mentally and are used to check for correctness and/or reasonableness of answers. These two skills are very different in that mental computation produces an answer that is right or wrong, whereas estimation may produce many different answers, all of which are reasonable and acceptable. A second difference is that mental computation requires that students have knowledge of basic math facts, whereas some types of estimation can be learned before learning basic facts. However, computational estimation is dependent on the ability to perform mental arithmetic. Students need to be taught when it is appropriate to use mental computation and when to use estimation.

Teaching Computational Estimation

Computational estimation is a set of skills that can be taught. A comprehensive estimation curriculum in computation must address several areas:

1. Development of an awareness of and an appreciation for estimation.
2. Development of number sense.
3. Development of number concepts.
4. Development of estimation strategies.

(Reys, B.J., 1986)

An awareness of and appreciation for estimation are often difficult. Many students "feel funny" about not getting the "right" answer; these students must learn to develop a tolerance for error. Some suggestions for creating an appropriate climate for estimation are: use real-world applications extensively, emphasize situations where only an estimate is required, use easy examples in the early stages, accept a variety of answers, use oral work and group discussion, and help the students determine when estimation is appropriate.

The development of number sense can be associated with having the student recognize "sensible" or "reasonable" answers. Instruction in this area could begin by presenting the students with noncomputational situations in which they identify or supply a reasonable answer.

For example:

Sandra carried some books to school. She carried about _____ books?

5 50 500

Another activity which can be used to build number sense is presenting examples which have been solved and having students check to find unreasonable answers.

For example:

PICK OUT THE ANSWERS THAT DON'T MAKE SENSE

$$\begin{array}{r} 1. \quad 249 \\ \quad 27 \\ \hline \quad 416 \\ \quad 692 \end{array}$$

$$\begin{array}{r} 2. \quad 98 \\ \quad \times 24 \\ \hline 23,502 \end{array}$$

$$\begin{array}{r} 3. \quad \quad 22 \\ 6 \overline{)1320} \end{array}$$

$$\begin{array}{r} 4. \quad \quad 95 \\ 5 \overline{)447} \end{array}$$

(Trafton, P.R., 1986)

In developing number concepts, it has been suggested that teaching estimation skills prior to the teaching of the concept may enhance the student's understanding of that concept. Students' general lack of development of some number concepts can be noted by the fact that in a recent national study, it was found that 76 percent of 13-year-olds in the United States incorrectly estimated the sum of $12/13$ and $7/8$ (Carpenter, T.P., Corbitt, M.K., Kepner, H.S., Lindquist, M.M. and Reys R.E., 1981). Was it because they didn't know how to estimate, or because they really didn't understand what $12/13$ and $7/8$ represent? The answer is probably a combination of both. It is quite probable that teaching the correct algorithm could result in greater understanding of these mathematical concepts.

Research on computational estimation has suggested that performance can improve dramatically when specific estimation strategies are taught (Driscol, 1981). Four strategies suggested by Reys (1986) are: front-end, rounding, compatibles, and clustering or averaging.

Front-end estimation involves using the first digits (left-most digits) in a problem to make estimates. The front-end strategy is a two-step process:

estimate front-end and adjust for the other digits. This is a strategy that even young students can learn to use, and it is most appropriately used in addition, subtraction, and division.

Example 1:

\$1.26
4.79
.99
1.37

1. Front-end: total the front-end amounts
(1 + 4 + 1 = 6)
2. Adjust: group the cents
(.26 + .79 = \$1.00)
(.99 + .37 = \$1.00)

Thus: \$6.00 + \$2.00 + \$8.00



Initial
Estimate

Adjusted
Estimate

Final
Estimate

The rounding strategy is a powerful and efficient process for estimating the product of two multidigit factors. Rounding is a three-step process: rounding, computing with the rounded numbers, and adjustment. Students should be taught that the purpose of rounding is providing mentally manageable numbers. They need to learn to be flexible in their method of rounding--fitting it to the particular situation, numbers, and operations involved. It is important for students to realize that there is no right answer in estimation. Each rounding choice produces different, but reasonable, results. They should also learn to adjust answers for over and under-estimation.

The compatible numbers strategy refers to a set of numbers that are easy to manipulate mentally. The choice of a particular set of compatible numbers involves a flexible rounding process. This strategy is particularly helpful when estimating division problems.

For example:

Estimate: $7\overline{)3388}$

compatible sets

not compatible

$7\overline{)3500}$

$7\overline{)3000}$

$8\overline{)3200}$

$7\overline{)3300}$

$8\overline{)4000}$

$8\overline{)3400}$

The clustering or averaging strategy is well suited for problems when a group of numbers cluster around a common value. For example, estimate the total attendance:

Olympic Games (Sept. 1-7)

Monday 105,000

Tuesday 109,975

Wednesday 95,492

Thursday 103,823

Friday 100,224

Saturday 97,795

- (1) Since all the numbers are relatively close in value, the use of clustering is appropriate.

Estimate an "average" about 100,000.

- (2) Multiply the "average" by the number of values.

$$6 \times 100,000 = 600,000$$

Summary

Mental arithmetic and estimation have similarities and differences. Computational estimation is dependent on students being able to perform mental arithmetic, and mental arithmetic is dependent on the student's knowledge of basic facts. As has been described, estimation can be taught directly. As with all mathematics skills, three phases should be included in the instructional process: direct instruction, practice, and testing.

**SOLVING
WORD
PROBLEMS**

SOLVING WORD PROBLEMS

Problem solving involves applying mathematics concepts and skills to situations that arise in day-to-day living. Problem solving requires the pupil to devise a plan for the solution and to carry out the plan. Two general types of problems are found in problem solving: routine problems, which involves choosing the correct mathematical operation and then doing the calculation, and nonroutine problems, which require the use of higher order thinking skills such as using deductive and inductive reasoning to draw conclusions. In this handbook we discuss only the solving of routine word problems.

Research studies indicate that problem-solving skills (using the processes and strategies of problem-solving) can be taught directly. Further, the teaching of problem solving skills is most successful when the problems to be solved by students are related to the world around them.

An important consideration in selecting word problems for students to solve is the level of reading skills required to solve the problem. Enright (1983) has noted that word problems in most math textbooks require students to read at either the instructional or frustrational level; thus, many students may experience difficulty in reading the problem prior to trying to solve it. Enright further suggests that word problems should be read at the independent level (i.e., understanding 90-95 percent of the words instantly) in order for problem-solving skills to be learned.

The process of problem solving can be taught as follows:

- READ (interpreting the problem)
- PLAN (devise a plan for solving the problem)
- COMPUTE (carrying out the plan)
- CHECK THE ANSWER (review the reasoning process and the result)

Read

In reading the problem, students need to determine the question being asked. It is often a good idea to have students underline the question being asked. You may also wish to have the students write the problem in their own words. In addition, they need to learn to organize the facts that are stated in the problem. Generally, it is helpful to have the students make a list of these facts. The next step is to have the students determine which of the facts apply to the problem; they should be encouraged to mark out the facts that are not needed to solve the problem (eliminate extraneous information).

Plan

This step in the process of problem solving is the difficult one. It is not possible to provide students with a quick and easy way to determine the plan to be used. Students should be encouraged to use the following strategies:

- * choose an operation
- * work backward
- * guess and check
- * look for a pattern
- * use a model

- * make a drawing
- * act it out
- * solve a simpler related problem
- * check hidden assumptions

Compute

This is the step in the problem solving process in which the actual calculation is performed. Some students who are good problem solvers have trouble with this step due to poor computational skills. The use of a calculator may be considered.

Check the Answer

Many students overlook checking their answers to determine if the answer is reasonable and if it makes sense in the context of the problem. Thus, students should be encouraged to review their answers. Estimation should be encouraged. A strategy that forces students to rethink their process for solving the problem is to have the students write the answer in a complete sentence. This written answer should include the main elements of the problem statement that was solved, as well as the solution to the problem.

Summary

Teaching students to read, plan, compute, and check the answer to a problem in a systematic manner will enhance their ability to solve problems. It is important to present problems that are easily read by the students. It is suggested that students be encouraged to generalize problem-solving skills to other subject areas. Activities which can be used to teach problem-solving skills are: have students generate their own problems, use problems from old math textbooks, and conduct a math bee in which the teacher reads the word problem and students respond by restating the question to be asked or determining the arithmetic operation to be used.

**TRAINING
OF
TRAINERS
MODEL**

TRAINING FOR IN-SERVICE PRESENTATION

The material in this section has been adapted directly from:

TRAINING THE OCCASIONAL TRAINER

Completed by the
Trade Task Force on Training the Occasional Trainer

Project Manager: Marcus Wiseman
Assessment and Field Support Program
Oak Ridge Associated University
Oak Ridge, Tennessee

December 1983

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INTRODUCTION

Teachers who teach children might assume that providing instruction to adults and peers should be easy. However, most teachers are well aware that "instructing" requires more than simply presenting material, regardless of whether you are teaching children or peers. Presenting material requires the art of developing rapport and understanding with the audience.

This section of the handbook is designed to help the occasional trainer develop the skills needed when instructing adults. Learning to define your audience, generating instructional objectives and training plans, and developing a personal training style are addressed in this section.

DEFINING YOUR AUDIENCE

Defining your audience is the first step in designing and preparing for a presentation. Whether this is a workshop to be conducted for one hour or several days, and even if you are an experienced presenter, this is an essential step. A class is a contract between you and your audience to increase their knowledge or skills. Therefore, if your message is to have the maximum impact, you need to identify their needs and understand why they are in the class.

The penalties for not defining your audience are many: If your presentation is too sophisticated or too complex, the learners will be bored, intimidated, or unable to learn. If your presentation is too elementary, the learners will be bored, feel put down, or have too little to learn. The rewards for being on target with your presentation are also many: The learners will be interested; they will develop new skills and knowledge; and you will be regarded as competent, knowledgeable, and certainly worth listening to. The focus of successful training is learner-oriented to maximize the transferral of your skills and knowledge to others.

How do you define your audience? Included here are questions to ask yourself before you design the class. The questions are divided into two areas: Analyze Need: "Who" and Analyze Need: "Why".

Analyze Need: Who?

How many participants will there be? It is essential to know the number of participants in order to plan your room size, the quantity of materials and

equipment that will be needed, and the kinds of training methods you will want to use.

What is the mix of the participants? How long have they been teaching? What grades or subject areas do they represent? Knowing the mix of the participants will give you information about the range of their experiences and offer clues about their particular orientation to the topic.

What is their knowledge and skill level? This information will help you plan how sophisticated, or how elementary to make your presentation. It will also guide the kinds of resources and the length of time you will want to spend on a subtopic.

How do they learn best? Many people prefer to learn the message through visual media (flipcharts, graphs, films), others prefer to hear the message (lectures, audiotapes), while still others prefer to learn by doing (demonstrations, practice, or equipment). Determining these preferred learning modes ahead of time will ensure that you have the kinds of materials that will maximize the learning experience.

What is their attitude about the topic? The answer to this question will clarify how you need to work with the trainees. If they have come voluntarily and are eager to learn, then you will have an ideal environment for instruction. Involuntary learners are not necessarily hostile. Sometimes they are unclear about why they are in the session or about what is expected of them. You will need to give clear statements about the purpose and contents of the presentation and have them identify their expectations. If you suspect negative feelings, you don't have an impossible situation on your hands. However, you will need to spend extra time on the purposes, structure, and

expectations, both yours and theirs. Negative attitudes can be softened by encouraging learners to participate more actively in the sessions.

Analyze Need: Why?

What are the expectations? You need to know the expectations of the teachers and the principal. If you can have participants identify their expectations at the beginning of the session, then tell them which ones will be met during the presentation and which ones will have to be met by other means, you are more likely to have a successful training session.

What are your expectations? As the instructor, you bring a wealth of knowledge and skills to the class, as well as your own interests. When they match the needs and expectations of the group, your presentation will be more successful.

Does the class description define the class content? A well-written course description that people have read will eliminate many distracting and confusing expectations.

How will they use this information? Is this information that they will be expected to apply, or is it background information for later use? If the desired result is for understanding the concept of an idea, then you will need to focus your presentation to meet the need. If the desired result is behavioral change, then you will want to plan for more practice and less theory.

Questions for the Trainer

In addition to the questions you need to be asking about the participants and why they are present, there are three questions you need to ask yourself.

What are my biases? As instructors, we have opinions and biases about what is important about a particular topic and how others should feel about it. Knowing those biases will help you understand the thrust of your presentation and help you bridge any differences that might arise in the session itself.

What will I learn from this session? It is an old saying that instructors teach what they love or what they need to learn. As instructors, we are also learners; to the extent that you can learn from your participants and from the content, the more lively the topic will be for you and the group.

Why do I want to do this? For some persons, the opportunity to be the resident expert is a motivation. For others, it is increased recognition by their peers or supervisors. Perhaps it is a real love for guiding people into new areas of knowledge. Rarely do occasional trainers get an increase in pay. Assessing your own motivations can help you increase the effectiveness of your class.

Summary

The intent of these questions is not to intimidate you, but rather to ensure the success of the presentation for you and the participants. Many of these questions can be answered intuitively; others might take more formal assessing. How do you find the answers to help you define your audience? You can get information in several ways:

Survey the participants before they come to the session. You can use a formal survey that asks them to rank their skill and knowledge abilities, or you can informally ask a random sample of participants about their skills and knowledge levels and expectations.

Survey the participants' supervisor.

Do some analyzing on your own. Reflect on your guesses of who will be attending and why they will be there. Use your own past experience in learning the topic to analyze how you learned and whether you think that the participants will learn in the same way or differently.

ANALYZING TRAINING NEEDS

Analyze Need: "Who?"

How many trainees will there be?

What is the mix of trainees?

men-women?

grade level taught?

age range?

number of years in teaching?

What is their special interest area?

Does the course description define the participants?

What is their knowledge and skill level?

How does this group of participants learn best?

What do they already know about the topic?

What is their skill level in this topic area?

What can they handle in terms of experimental training?

What is their attitude?

What are their attitudes toward the topic?

Were they sent, or did they come voluntarily?

Are they hostile? friendly? indifferent?

Are they willing to learn? Are they daring you to try to instruct them?

Are there words or jargon that will offend or distract?

Are there issues that will push their buttons?

Are they interested?

Are they intimidated or fearful of the topic?

Analyze Need: "Why?"

What are the expectations?

What are the job/performance expectations?

What are the gaps between reality and expectations (i.e.,

what they will learn in the course versus what they want to learn)?

What are your expectations? Theirs? Their supervisors?

Who put me up to this? What do they want the trainees to learn?

Is it a requirement that they know this information?

What does this group need?

DEVELOPING A TRAINING PLAN

Once the instructional objectives have been developed, your effectiveness as a trainer will increase greatly if you take the time to write a training plan to show how the objectives will be achieved. A training plan is not a script. It is an outline of what is to occur during the presentation. The outline includes the instructional objectives, content, instructional methods, materials and equipment needed, participant activities, and evaluation measures.

The training plan does take time. However, this step should not be bypassed. The writing of a training plan will force you as the trainer to sit down and prepare for the presentation. This preparation will help overcome the anxiety you may feel about facing a group. The training plan is particularly useful for occasional trainers as it increases their confidence in their ability to be effective presenters.

Where to Begin

The first step is to size up the group you will be facing or "Define Your Audience." As the trainer, you are in the best position to determine the appropriate methods of assessing the background of the group to receive instruction. Some suggestions include:

- o Have the attendees complete some form of preattendance work, such as a questionnaire or self-rating checklist.
- o Use a pretest of training materials.

Organizing and Preparing the Training Plan

Trainers are usually selected to conduct presentations because of their knowledge or understanding of the material to be presented. It might be helpful to update yourself on the topic to be covered and review materials that you will be using.

Next, you will need to write your instructional objectives and select the content to support these objectives. Remember to give some thought to the make-up and needs of your audience.

Mindful of these considerations, let us get to the actual material needed to complete a training plan. Collectively, this information will make for a well-prepared presentation. The presenter will be assured of having the needed materials and equipment for the presentation at hand. This aids in presenting the subject matter in a complete and logical order.

Examine the sample training plan. It serves as a blueprint to which the presenter can refer as the presentation progresses. Notice the following items and where they are located on the sample training plan:

Subject title

Training segments

Materials and equipment

Instructional objectives

Content

Methods/trainer activities

Participant activities/evaluation

Time allotted for each segment to be taught (TA)

Now let's examine the following items noted.

TRAINING PLAN

COURSE:

SUBJECT:

TRAINING SEGMENTS COVERED	MATERIALS AND EQUIPMENT	INSTRUCTIONAL OBJECTIVES	CONTENT	METHODS/ TRAINER ACTIVITIES	PARTICIPANT ACTIVITIES EVALUATION	TA
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Content

Clearly, this is only a broad outline that covers the main points of the materials. The outline should contain enough information for the presenter to feel comfortably informed.

As an instructor, you should know how to present the material in a way that is meaningful to the participant, rather than simply convenient for you. This is important because the purpose of the presentation is to encourage learning by the participant. One helpful way to accomplish this is to arrange the presentation content in proper sequence. Approach instructional sequencing by asking these questions:

- o How will the participant use this?
- o Does this emerge logically from something the participant already knows or can do?
- o Does this prepare the participant for further learning?

There are several approaches to instructional sequencing. Following a few general rules will be helpful.

1. Instruction should proceed from the general to the specific.
This technique gives the participant the big picture first, then fills in the details.
2. Teach those skills that are used most often first.
3. Put high-interest materials at the beginning, which is an excellent way to maintain motivation.
4. Move from the familiar to the unknown to let old material serve as a basis for new material.
5. Move from the whole to the parts.
6. Move from the concrete to the abstract.

Methods/Trainer Activities

The presenter should vary the methods used for each part of the training. Don't just lecture! Various trainer methods will be presented in a later section.

Included under trainer activities are the "housekeeping" activities, such as "pass out handouts." Listing these activities affords the trainer the confidence of remembering just when materials are to be distributed.

Participant Activities/Evaluation

This section of the training plan describes what the participants will do during each segment of the presentation. Activities measuring participant performance, such as observation, simulations, and tests, are also listed. Evaluation of learning is an ongoing process. It is the systematic process of assessing whether instructional objectives are being met. The written test is only one means of measurement--performance tests, oral questions, and observation of participants at work on a task also serve as measuring tools.

Time Allotted (TA)

The time allotted will simply keep you aware of the amount of time being spent on each segment. It is not necessary to adhere rigidly to those times. If more or less time is needed, that is fine. These time allotments are handy when adjustments need to be made. When developing your training plan, inflate your time allotment by 10 percent to allow enough time for completion.

Summary

A training plan is an outline of the learning experience you will offer. It must reflect the level of importance of the learning to be achieved. It is the basic organizational unit and schedule of events for the presentation.

A training plan will enable the trainer to organize and present the materials with confidence. There is security in knowing that the information is in an organized, logical order, that meets the instructional objectives.

Every training plan should at least include the learning objectives, the content outline, the training methods to be used, and the participant activities/evaluation. They should appear on the training plan and in a form most convenient for the presenter to use as a reference.

Training Methods

The objectives and the subject matter have been organized. Now the trainer needs to select the most appropriate training (instructional) methods to present the subject matter and accomplish those objectives.

Training methods are the approaches, strategies, and procedures that are involved in the learning process. Training will be more effective if the trainer is careful to vary the methods used during the presentation.

The instructor's job is to select an organized set of activities that will result in meaningful learning experiences for the participants. The instructor should choose methods of instruction in terms of the participants' activities as learners. To choose appropriate methods, the instructor must keep in mind the ways in which adults learn:

- o By doing
- o By discussing
- o By listening
- o By observing
- o By participating

Several different methods of instruction might be used in a single presentation. Combinations of instructional methods provide variety for the participants, which helps to hold their attention and maintain their interest.

There are many popular training methods. This section will address these four: lecture, guided discussion, demonstration, and practice.

1. Lecture Method

The lecture method usually refers to a formal or informal presentation of information, concepts, or principles by a single individual.

Advantages:

1. It permits the presentation of many ideas in a short time.
2. Lecture is particularly suitable for introducing a subject or presenting basic information.

Disadvantages:

1. Lecture is not appropriate for certain types of learning, i.e., skills.
2. Feedback is limited, especially in formal lectures. Much information is presented in a short period of time, and trainers may not be able to determine what has been learned.

3. It is a convenient method for instructing large groups.

3. Learning is an active process; the lecture method tends to foster passivity and dependence on the instructor.

Techniques for a Successful Lecture

- o Lectures should always be supported with visual aids such as chalkboards, flipcharts, handouts, slides, and overhead projectors.
- o Determine the purpose of the presentation, and organize it by choosing the main points and arranging them in logical order.
- o Depending on the participants' knowledge of the subject matter, decide how to make the points clear. Choose examples--short, colorful, occasionally humorous--that relate to the experiences of the participants. Strong organization and clear verbal and visual support help participants understand and retain the material.
- o Use audio-visual aids to accompany the talk, but only materials that are relevant and large enough to be seen by all the participants. Keep the visual material as simple and clear as possible.

2. Guided Discussion Method

The guided discussion method is an instructor-controlled, interactive process of meeting instructional objectives by sharing information and participant experiences. In a guided discussion, the instructor asks questions and summarizes concepts and principles learned, but tries not to dominate the discussion. Unlike the lecture, participants are active. They are encouraged to learn about the subject matter by actively sharing

ideas, knowledge, and opinions. The instructor must carefully plan the lesson to attain the desired learning outcomes. This goal is accomplished by guiding the discussion with appropriate questions and periodic summaries. A guided discussion can follow a lecture.

Advantages:

1. Guided discussion develops knowledge and attitudes.
2. Guided discussion helps participants develop skill in reasoning and problem solving.
3. It accomplishes changes in attitude and behavior.

Disadvantages:

1. The size of the group limits the use of guided discussion.
A group of 5 to 20 works well.
2. Instructors must be willing to give up some of their control of the training session. The instructor must be able to follow the discussion, not lose patience with the participants, and, at the end of the session, summarize points made.
3. A guided discussion requires as much preparation time by the instructor as a lecture, but requires more time as a training activity. Generally it takes longer to cover an idea through guided discussion than with a lecture.

Techniques for a Successful Guided Discussion

Planning

- o Determine the basic purpose of the discussion.
- o Summarize what you know about the subject, and get more information if necessary.
- o List a few questions to start the discussion and to get major points across.
- o Determine whether visual materials will help the discussion, and, if so, arrange to have them.
- o Provide some means for listing the important points made during discussion, as well as questions left unresolved.

Initiating

- o Clarify the topic to be discussed and arouse interest. A short lecture, motion picture, videotape, or a description of a problem situation may be the basis for opening the discussion.
- o Phrase your questions to require more than just a "yes" or "no" answer. At the same time, make the questions concrete enough to permit the participants to organize their thinking about them and respond within a few minutes.
- o Be prepared to wait for someone to respond.
- o Materials such as a series of case problems, questions, and factual data can stimulate meaningful discussion.

Guiding

- o Encourage the participants to do most of the talking until you have an idea of how much they know and where you, as the resource person, can best make your contribution.

- o Ensure that each participant who wishes to speak gets the opportunity and that no one dominates the discussion.
- o Repeat some of the ideas to emphasize the vital points that have been made.
- o When necessary, offer information, correct misunderstandings, and bring out additional points of view.
- o Keep the discussion on the subject. One way to get back on course is to summarize briefly what has been said up to that point, then ask questions to go on from there.
- o Promote exchange of opinions among participants.
- o If participants appear confused, summarize what has been covered up to that point, encouraging questions.
- o If disagreements arise, try by your own calm behavior to gain respectful, good-humored attention for all points of view. A trainer should not become involved in the argument itself.

Ending

- o Watch for signs of restlessness, and end the discussion while interest is still high.
- o Summarize the major points that have been made, or have a participant summarize. If major points have not been adequately covered in the discussion, include those in your summary.

3. Demonstration Method

The demonstration method is a practical, step-by-step performance by the instructor of a process, procedure, or other activity with a detailed explanation of each step. A demonstration should be accompanied or immediately followed by having participants practice the activity or skill

being demonstrated.

Advantages

1. A demonstration is especially helpful in technical and skill training, because it is the link between explanation and practice.
2. By watching a demonstration and doing the activity themselves, trainees are more likely to remember than if they simply hear an explanation or watch a demonstration.
3. The practice also offers trainees the opportunity to be physically active after periods of passive sitting and listening.

Disadvantages

1. A good demonstration requires a great deal of time for proper preparation.
2. A demonstration of a skill for imitation by a trainee should not last too long. The only way to master a skill is by doing, not by seeing.
3. Class time must be sufficient for all necessary stages of the demonstration. Avoid presenting something new before the preceding point is assimilated. People can only watch one thing at a time.

Techniques for a Successful Demonstration

- o Break down the activity or skill into steps in logical order.
- o Plan sufficient class time to permit an introduction, step-by-step demonstration and explanation, thorough practice by the trainees with guidance by the trainer, and final discussion and review.
- o Arrange the training area so that all trainees can clearly see and hear the demonstration.
- o Have the necessary materials in order and within easy reach.
- o Tell briefly what you are going to demonstrate, its value to the trainees, and what in particular the trainees should watch for.
- o Have the trainees try what you have demonstrated either after each step or at the end of your demonstration. Allow sufficient practice time so that all trainees may receive guidance. Encourage trainees who know the skill or learn it quickly to help others.
- o At the end of the practice time, emphasize again how trainees may use the new activity or skill on their jobs. Review the steps demonstrated. Describe the equipment needed, and suggest possible substitutes. Suggest resources for further study.

4. Practice Methods

Practice methods provide trainees with experience in applying new knowledge, skill, and attitudes under conditions similar to work situation experience. The practice method should always include immediate feedback, allowing the trainee to correct mistakes.

Many different types of activities have been developed using the practice method. Two of the most common practice activities are role playing and case studies.

Role-Play

In a role-play, the trainees act out a situation based on real life. No acting ability is necessary because of the real-life quality of the situation. Trainees role-play the attitudes and behaviors involved in carrying out a task or job responsibility.

Advantages:

1. It is highly participatory for the trainees. Each trainee assumes a role and acts out an attitude or situation relevant to his/her job.
2. It is useful in improving skills.
3. It helps trainees become aware of the reasons behind behavior and of the effects of their own behavior on others. This awareness causes the role-players and

Disadvantages:

1. It must be restricted to clear-cut problem situations.
2. Role-play requires careful supervision and direction by the trainer.
3. Trainees may suffer anxiety from the anticipation of having to play a role.
4. Trainers may suffer anxiety from the participation of having to

observers to experience more
closely the feelings and
reactions connected with the
situation.

play a role.

4. It is a more vivid experience
than merely taking about a
problem.
5. Role-playing lends reality to
any theory that has been presented.

5. It takes a great deal of training
time.

Techniques for a Successful Role-Play

- o Select a situation meaningful to the trainees.
- o Let trainees volunteer for the role-play, rather than suggesting that they take parts.
- o Give instructions to each player clearly detailing who he/she is and how he/she "feels" at the beginning of the role-play.
- o Give them a few minutes of privacy to get into the moods of their roles.
- o Allow the role-play to continue until real feelings develop among the players. This usually takes from three to seven minutes of action. The purpose of the role-play is to stimulate appreciation of the human behavior involved in the situation. It is not necessary to continue the action until the situation is resolved.
- o After the role-play, discuss what occurred.

Case Study Method

Case studies are written descriptions of real or imaginary situations. Presenting a case study gives trainees an opportunity to apply new knowledge. It stimulates discussion and participation. Case studies enable the trainee to develop skills in responding to various situations.

Case studies are particularly useful in helping trainees explore different ways to deal with typical problems. However, the case must be well prepared and focused on the problem areas.

The statement of the problem or case study should be brief and simply worded. The problem or case study should be one that involves decision and action. Finally, there should be definite instructions as to what is to be done. These instructions are usually in the form of questions addressed to the participants about how to solve the problem. Guide the individuals to consider each of the following steps:

1. Look for the real problem. By getting answers to questions such as who, what, when, where, and why, you can pinpoint the problem and its causes.
2. Gather all the facts. Eliminate guesswork and opinions.
3. Evaluate the facts.
4. Develop possible solutions.
5. Select the best solutions.

Although all steps may not apply to a particular problem or case study, the individuals should be aware of the steps.

Training Methods

Handout No.1

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THE DELIVERY OF TRAINING
Participant Involvement Continuum

Medium

Simulations

Role-Play

Case Study

Demonstration

Guided Discussion

Informal Lecture

Low

LECTURE METHOD
(Informal Lecture)

Advantages

1. Present many ideas in short period of time.
2. Suitable for introducing a subject or giving instructions.
3. Convenient method for instructing large group.
4. Useful for giving directions and summarizing main points.

Limitations

1. Not appropriate for skill training.
2. Difficult to evaluate trainee progress.
3. Low trainee retention.

• When to Use

1. Presenting theory or ideas.
2. Introducing a subject or giving instructions (used in combination with other methods).
3. Have little time or resources.
4. Not aiming to build skills.
5. Have large group.

GUIDED DISCUSSION

Advantages	Limitations
1. Useful for knowledge and attitude objectives.	1. Group size limited.
2. Develops skills in group reasoning and problem-solving.	2. Instructor must give up some control.
3. Reviews material already	3. Needs a large amount of time.
4. Can stimulate critical thinking.	4. Discussion may be dominated by a few.

When to Use

1. Changes in attitudes needed.
2. Gaining of understanding of differing points of view.
3. Attainment of group consensus desired.
4. Sharing of trainees' experiences sought.

HOW TO USE QUESTIONS EFFECTIVELY

To avoid long, structured monologues, the trainer can simply use questions. In fact, he/she can recede into the background almost completely if the right questions are used.

Types of Questions

Closed questions. A question that can be answered adequately in a few words, i.e., yes/no or straight facts. (Naturally these questions fall short in stimulating a discussion.)

Open questions. A question that requires more than a few words for an adequate answer. (These will stimulate participation and discussion.)

Characteristics of Good Questions

- o The answers are more than one word.
- o The answers are not obvious.
- o The question is concisely worded.
- o The question is easily understood.
- o Use of "Why, What, When, Where, Who, and How" appear in all open questions.
- o Each question requires thought before answering.
- o Each question relates to the lesson being taught.
- o Finally, the questions are prepared in advance.

Using Questions

- o Prepare--Plan your questions.
- o Ask the question--This starts the learner's learning process.
- o Wait for the answer. New trainers have very little tolerance for silence.
- o Ask someone by name--This might be necessary to break the ice.
- o Listen to all answers--Don't cut anyone off. Get all the answers, then sort them out. (You might cut them off forever.)
- o Review, Reinforce, Repeat--the 3Rs (used in all training).

Questions are a major function in getting total class involvement. Trainee participation is never easy. The trainer must have faith in the concept and have the will to develop participation. Otherwise, the inevitable result will be more lectures and more speeches, courtesy of the trainer.

DEMONSTRATION

Advantages

1. High trainee involvement.
2. Calls attention to detail.
3. Good for trainee retention.
4. Gives immediate feedback to trainee.

Limitations

1. Takes more training time.
2. Group size limited.
3. Often lasts too long.
4. Most useful for theory training.

When to Use

1. Want to transmit hands-on skills.
2. Want to see immediate results.
3. Concerned with direct application of learning.

ROLE-PLAY

Advantages

1. Highly participative for trainee.
2. Improves people-to-people skills being exhibited.
3. Lends reality to theory learning.

Limitations

1. Takes lots of training time.
2. Possibility of strong emotions.
3. Must be kept within very strict boundaries.

When to Use

1. To build skills in interpersonal relationships.
2. Concerned with direct application of knowledge.
3. To improve supervisory skills.
4. Effect changes in attitudes.

GROUP DYNAMICS

Each group you face as a trainer develops a specific identity, evolves through different stages, and contains individual members who play definable roles. It is important for the trainer, especially the occasional trainer, to understand that the changes in the mood, level, and type of participation by group members are natural developments and to be expected. This process is known as group dynamics. This section describes the stages that most groups pass through and the various roles that the trainer and trainees play.

Group Stages and Roles

People in groups generally behave differently from the way they behave alone or with just one other person. Generally, the less the members of a group know one another, the more different their feelings and behavior will be from their private ones. Most of us have a style of talking with a group of strangers that differs from our style with just one new acquaintance. Our differences in behavior and feelings are not confined to conversations only. Think of the last time you silently sweated through an elevator ride with six strangers.

Social scientists have studied small groups extensively and have documented the fact that these groups tend to move through certain characteristics and predictable stages. At each stage, certain behaviors occur. They have also observed that certain roles or functions often are assumed or played out by members in a group. If trainers are aware of these behaviors of groups and the typical roles that are played by members of a group, they are in

a better position to direct the group in ways that will achieve the training objectives.

Group Stages

There are many theories of group development (i.e., of the stages through which groups move and of the characteristics or issues that typically dominate each stage). One model can be described as a complex three-stage model: (1) orientation stage, (2) integration stage (conflict/cohesiveness), and (3) work stage. The first stage is concerned with inclusion needs. The second stage deals with control--who has power, and how is it shared? The third stage finds the group dealing with the level of trust between members. The group characteristics to be discussed may occur at any of the stages, but generally occur as indicated.

Stage One: Orientation

A sizing up or assessment process takes place at the beginning of a group. Individuals will be concerned with whether they can depend on the trainer and whether the training experience will satisfy their individual needs and goals. "Will I get the knowledge, skill, and experience in this session that I want?"

Group members will be concerned with the group norms (i.e., what behavior is considered acceptable by the trainer and by the other trainees?). "Can I be expressive and animated in this group, or must I just speak quietly and deliberately?" "Will this be a monologue by the trainer or a series of dialogues between the trainer and individuals or a verbal free-for-all?" "Can the trainees use humor and, if so, what kind?"

Stage Two: Integration

At this stage, the group begins to deal with leadership issues. How will the trainer lead, autocratically or democratically? Is the trainer really the leader of the session? How much of a leadership role can class members assume vis-a-vis the trainer and each other? Related to this, of course, are power issues. "How much power do I as a member bring with me by virtue of my job status, my knowledge of the subject matter, or my verbal and analytical skills?" "How much power will others and the trainer attribute to me, and how much will I attribute to them?" Sorting out these issues and the leadership and power relationship in the class will engender conflict and, at times, open confrontation with the trainer or other trainees.

In many small groups lasting a day or longer, some of these issues will arise and must reach a degree of resolution for group cohesiveness to develop. As they are resolved, cohesiveness increases:

1. The group will express a great deal of solidarity. Members will comment, for example, "We did a good job on that."
2. The members will actively defend the sense of solidarity from internal threats (e.g., disruptive conflict) or external threats.
3. Members will assist each other in ways they perceive as needed and will actively play out various roles in the group.
4. Members will strongly defend the group norms that have been established and will not allow them to be broken or changed.

Stage Three: Work Stage

If a group develops to this point, a significant level of trust has developed between group members. This trust makes possible the self-disclosure

and feedback that characterize the third stage. Self-disclosure refers to an individual's sharing information about himself or herself. Furthermore, this information is the kind of information that a person would not share with everyone. Feedback refers to the process of sharing with another person how one thinks, feels, and behaves with respect to some perceived aspect of the other person. This last stage of group development optimizes the possibility of effective feedback, feedback that the individual can genuinely "hear" and act upon.

Group Member Roles

Group member roles can be categorized into four broad types of activities or ways of behaving that help a group accomplish its goals. One broad category helps a collection of individuals become, continue, and grow as a group. These are "maintenance roles" and include such behaviors as:

- o Encouraging, keeping motivation by commending other's contributions.
- o Gatekeeping, ensuring that everyone has a chance to contribute.
- o Harmonizing, mediating others' differences by listening and expressing group feelings.

Another category, that of task roles, helps the group accomplish its training tasks:

- o Seeking information for input or clarification of the task.
- o Giving information, offering facts or background on issues.
- o Summarizing, i.e., pulling things together, recapping, adding perspective.
- o Testing workability or seeking consensus.

A third category contributes to both the task and maintenance dimensions of a group. These combined roles include:

- o Evaluating, assessing the usefulness of input made by group members.
- o Diagnosing, finding which areas of a task are in need of attention.
- o Testing for consensus, surveying members to determine areas of agreement.
- o Mediating, intervening when conflicts arise to resolve conflict.
- o Relieving tension, helping the group to work through difficult times of conflict.

A fourth general type of activity includes those roles defined as "non-group roles." These are attention-getting devices that hinder accomplishment of tasks and typically include the following:

- o Dominator: interrupts, give excessive advice.
- o Blocker: makes judgmental comments, lacks toleration of individual differences.
- o Avoider: does not deal with the present situation, changes the subject, doesn't follow through on assignments.

IMPLICATIONS OF GROUP DYNAMICS FOR EFFECTIVE DESIGN OF TRAINING

The main implication of group dynamics for effective design of training is that knowledge of group stages and of roles in a group allows the trainer to mesh group dynamics with attainment of the instructional objectives. There are two main points to be made: first, small-group dynamics provide limits that a trainer cannot really change. If the trainer understands these limits, he or she can avoid risky or disastrous activities and will be less likely to be

"surprised" by unanticipated developments. Second, knowledge about small-group dynamics can help the trainer plan more effectively and exercise "control" over what will occur.

For example, the trainer cannot escape, at the beginning of a group session, the characteristics of stage one. The use of "icebreakers" or another activity to open a session is not an accident, but is designed to relieve trainee anxiety. (The trainer is also nervous in the initial stage of a group and wants to know how he/she will be received by the group, just as much as other members of the group are concerned about their relationship to him/her and to each others.) In the first stage, ground rules are being formed, and the instructor's role is critical here. Since group cohesiveness has not been established, to run a high-risk role-playing exercise at the inception of a group is dangerous. If someone is perceived as having been badly "burned" at the beginning of a group, the entire training session can be damaged.

In a day-long session, some group members will challenge the trainer and each other (leadership issues of stage two). The trainer should also expect some conflict in a day-long session. These events and others like them will occur and must be expected and planned. How the trainer diagnoses and handles the first and second stages will determine how the group will communicate.

A training group may not display all the characteristics of each of the stages that a "mature" working group displays as it progressed to its mature level of functioning. In fact, most training sessions will not reach the final stage of maturity; they are too short, dominated by lecture, or technical in content, etc. However, a two-hour session provides ample time for the first-stage characteristics to emerge, and a day-long session will ordinarily find a group manifesting stage one and two behaviors.

POINTS TO REMEMBER: GROUP MEMBER ROLES

1. Any member of the group can play several roles, depending upon the circumstances.
2. Nongroup role behavior isn't always "bad."
 - o It means someone's needs aren't being met.
 - o Someone may simply be saying, "Stop and let me make my point!"
 - o Dealing with nongroup behavior can encourage the group to grow.
3. To get the job done, the group needs both task and maintenance role behavior.
4. As trainer, you may have to fill roles not assumed by others.

Summary

According to one model of group dynamics, groups generally develop through three stages. In the first, orientation, group members are typically concerned with determining what behavior will be acceptable in this group situation. The second stage, integration, involves the development of leadership and power relationships in the group. In the work stage, if it is reached, group members are willing to share information about themselves, and effective feedback is truly possible. Group member roles that help a group accomplish its goals are categorized as maintenance roles (helping the individuals develop as a group), task roles (helping the group accomplish its training tasks), and combined roles (contributing to both group maintenance and task accomplishment). Non-group roles involve attention-getting devices, which hinder accomplishment of tasks.

This section has introduced the concept of group dynamics and argues that the trainer should be aware of these dynamics when designing and implementing a training session. Some of the dynamics of a group are beyond the trainer's ability to predict and consider in designing a training session. Others are beyond the trainer's ability to control, even if he or she can predict them. However, the trainer does have enough knowledge and enough capability to work with the trainees to inhibit or encourage certain events and to mesh other elements of the training process (style, content, length, etc.) with group dynamics to foster the achievement of the instructional objectives.

This section on group process is not intended to "mystify" the training process, but to introduce the occasional trainer to one of the key elements of good training design and implementation. The occasional trainer is not expected to become an expert in group processes.

GROUP DYNAMICS

THE "BLOCKER"

Argumentative

TRY THIS:

- o Ask participant to summarize the position he or she disagrees with.

- o Change the subject.

- o Let the group handle this participant.

Off the subject

- o "That's an interesting point, but let's hold it until later on."

- o Direct participant's attention to topic on board or visual.

Side conversations

- o Stop talking and wait for side conversation to end.

- o Stand behind the participants who are talking.

- o Ask a direct question of one of the talkers.

THE "DOMINATOR"

TRY THIS:

- o Interrupt tactfully with a question or summarizing statement.

- o When the talker pauses, rephrase one of his/her statements, and move on.

- o Allow the group to cut this participant off.

THE "AVOIDER"

TRY THIS:

- o When asking a question, make eye contact with this participant.
- o Involve this participant in small group work, and ask him or her to report back.
- o Ask a direct question of this person.

TRAINER STYLES

For many occasional trainers, the thought of standing up in front of a group is not met with great enthusiasm for reasons such as a lack of instructor training or a feeling of not being adequately equipped to conduct training. This reluctance might be due to a lack to understanding of "training " itself. Yet we all do training, whether teaching a child, helping a neighbor with a household repair problem, or advising a friend. These things come naturally to all of us. Many of the same skills used in these examples can be applied to responsibilities as a trainer.

Good instructors plan their training and use effective methods and techniques that involve the learners. Good instructors also look closely at their assumptions about people and how they learn. They are aware of their interpersonal style, the impact of their behavior on others, and the strengths and weaknesses of their individual style. Awareness of these characteristics helps trainers improve their effectiveness.

Everyone holds basic assumptions about people that guide their interaction and behavior patterns. These hopes and beliefs direct the way we act around others, but often they have never been consciously or critically examined.

These beliefs are often held so strongly that they are not recognized as personal beliefs, but merely as "the truth." A typical assumption of this type is that people must be coerced and directed toward the achievement of learning objectives. In addition to our beliefs about the nature of people and of learning, we also have behavior patterns that may be so habitual that we are unaware of them. These unconscious behaviors can lead to situations in which trainers believe they are being attentive and supportive, totally unaware of the frowns or yawns that indicate disagreement or lack of interest.

We need to know how our behaviors affect others and to check how well our behavior reflects our own beliefs and assumptions about how people learn. This self-examination allows us to build on our strengths and eliminate or lessen any weaknesses. These changes will also improve the training session and make the job of training a more comfortable one.

ACHIEVING SELF-KNOWLEDGE

There are several ways occasional trainers can learn about themselves and their impact on a group of trainees. After making a training presentation, occasional trainers can solicit direct verbal or written feedback about themselves and their performance as trainers. While not as informative, a self-critique might be less threatening than asking an actual group of trainees for the same information.

Educational Assumptions

Another dimension of trainer style is the trainer's assumptions about how people learn and how people ought to teach. Most persons, even most teaching Ph.D.s, have given little serious thought to questions about the philosophy and practice of education. Most of us tend to model our teaching/learning theory and behavior on our past experience, with little reflection. We believe that teaching and learning are quite properly done the way our parents taught us or the way we were taught by a favorite teacher or professor. Just as greater knowledge of our interpersonal style can be useful for the trainer, so can an exploration of our educational philosophy. Such an exploration would be particularly helpful if done in the light of some alternative assumptions about teaching and learning.

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